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Finite quotients of Calabi-Yau threefolds

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INTRODUCTION

This thesis is devoted to the study of arithmetic and geometric properties of Calabi-Yau manifolds obtained as a quotient of Calabi-Yau threefold by an action of a finite group. We compute Hodge numbers of Calabi-Yau threefolds of Borcea-Voisin type using stringy Euler characteristic and Chen-Ruan cohomology. We construct new Calabi-Yau threefolds as a quotient of double octic threefold by an automorphism subgroup. Moreover we give a sufficient condition for rigidity of resulting manifolds. Consequently, by choosing examples of modular double octic threefolds satisfying this criterion, we obtain new modular rigid Calabi-Yau threefolds, which yield modular rigid realization of two cusp forms of weight four.

Our research is motivated by (hypothetical) applications of Calabi-Yau threefolds to physics in the so called string theory and inspiring with that theory, one of the most famous mathematical problem known as Mirror Symmetry Conjecture. Many families of Calabi-Yau threefolds which are important in this subject, were constructed via finite quotients of Calabi-Yau threefolds by a subgroup of automorphism group generated by non-symplectic automorphisms.

In Chapter 1 we introduce the notation and basic facts concerning Hodge theory of Kähler complex manifolds X. According to this theory, the following Hodge decomposition holds for each natural k :

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

with finite dimensional vector spaces $H^{p,q}(X)$, satisfying certain number of conditions. The dimensions $h^{p,q}(X) := \dim H^{p,q}(X)$, called the Hodge numbers of X, are one of the main invariants of complex manifolds, which may be visualized in the Hodge diamond. We shall give formulas for the Hodge numbers of a product of two manifolds (Künneth's formula) and for a blow-up of a manifold along a smooth submanifold. At the end of this chapter we briefly summarize basic results in deformation theory of compact complex manifolds that we shall need in our considerations.

The second Chapter is devoted to the study of Calabi-Yau manifolds, i.e. compact complex Kähler manifolds X having trivial canonical bundle and $H^i(X,\mathcal{O}_X) = 0$, for $0 < i < \dim X$. Triviality of the canonical bundle is equivalent to existence of a nowhere vanishing holomorphic form $\sigma \in H^0(X,\Omega^n_X)$, where Ω_X is the holomorphic cotangent bundle of X, and $n = \dim X$. We start with a description of the Hodge diamond of a Calabi-Yau manifold. Then we shall explain unobstructedness of the infinitesimal deformations and formulate the simplest version of the Mirror Symmetry Conjecture.

In the second part of this chapter we study $K3$ surfaces which are two dimensional Calabi-Yau manifolds. The study of non-symplectic involution on $K3$ surfaces using lattice theory, pioneered by V. V. Nikulin started new area of research, the classification of all non-symplectic automorphisms and description their singular loci is still incomplete and provides many examples of Calabi-Yau manifolds.

In Chapter three we focus on quotients of projective varieties X by an action of a finite group G acting on X. The resulting variety X/G possesses a very interesting (orbifold) structure, which is often singular. As we are mainly interested in Calabi-Yau manifolds, the natural question arises here: when does the given construction produces a smooth Calabi-Yau model of X/G? To this end we need a resolution of singularities which does not change a canonical bundle of X which are called crepant. Unfortunately this type of resolutions does not exist in every dimension and if it does, then not for all kinds of singularities. In dimensions two and three, by works of Klein and Roan such a resolution always exists. Moreover, from these results one can predict much deeper properties of crepant resolutions of finite quotients of $\mathbb{C}^n$ in higher dimensions, together with possible connection with other areas of mathematics it is known as McKay correspondence.

Next part of this chapter is devoted to a cohomology theory for finite quotients. W. Chen and Y. Ruan introduced

$$H^{i,j}_{\mathrm{orb}}(X/G) := \bigoplus_{[g]\in \mathrm{Conj}(G)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{\mathrm{C}(g)},$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G, $\mathrm{C}(g)$ is the centralizer of g, $\Lambda(g)$ denotes the set of irreducible connected components of the fixed points set of $g \in G$ and $\mathrm{age}(g)$ is an integer associated to a matrix corresponding to linearized action of g. Main advantage of this formula is that it provide a method of computing Hodge numbers of a crepant resolution of finite quotient, without giving an explicit construction of that resolution. In fact

$$\dim H^{i,j}_{\mathrm{orb}}(X/G) = h^{i,j}(\widetilde{X/G}),$$

where $\widetilde{X/G}$ is a crepant resolution of finite quotient X/G.

We end this chapter with a definition of stringy Euler characteristic introduced by physicists and motivated by string theory. This formula, just like Chen-Ruan cohomology, may be used to compute the Euler characteristic of a crepant resolution of finite quotients.

The fourth chapter aims to the Borcea-Voisin construction of Calabi-Yau threefolds which produces important examples in context of mirror symmetry. Main idea is to start with a $K3$ surface S and an elliptic curve E, both admitting non-symplectic involutions α_S and α_E, respectively, and consider the quotient $(S \times E)/(\alpha_S \times \alpha_E)$. The crepant resolution $\widetilde{(S \times E)/\langle\alpha_S \times \alpha_E\rangle}$ is a Calabi-Yau threefold with Hodge numbers

$$h^{1,1} = 11 + 5N' - N \quad \text{and} \quad h^{1,2} = 11 + 5N - N',$$

where N is the number of curves in $\mathrm{Fix}(\alpha_S)$ and N' denotes the sum of genera of all curves in fixed locus. Voisin used Nikulin's classification to show that constructed threefolds have a "mirror partner" for all but one of the examples with $N' \neq 0$.

A. Cattaneo and A. Garbagnati generalized the Borcea-Voisin construction allowing non-symplectic automorphism of a $K3$ surfaces of higher orders. They computed Hodge numbers of resulting threefolds by careful study of an explicit construction of a crepant resolution. We shall compute the Hodge numbers using orbifold cohomology and stringy Euler characteristic. These computations are the first main result of the present thesis. Our method based on orbifold formulas does not involve an explicit construction of a crepant resolution. We need only existence of such resolution, which is guaranteed by the Roan theorem. The main advantage of our approach is that the computations are carried out on $S \times E$ and they are much shorter and cleaner, as the construction of a crepant resolution proposed by Cattaneo and Garbagnati is technical and complicated.

In the last chapter we study the modularity of finite quotients of double octic Calabi-Yau threefolds. We start by the introducing general definitions, which are needed to study arithmetical concepts of projective varieties defined over $\mathbb{Q}$. We shall define a zeta function of a variety and state remarkable conjectures observed by A. Weil known as Weil Conjectures. In fact Weil proved the conjectures in the case of curves. The full generality was obtained by the works of B. Dwork, P. Deligne and A. Grothendieck.

The next section is devoted to étale cohomology developed by Grothendieck, and used in his proof of the part of Weil Conjectures. That tool together with Weil Conjectures allow to define L-series $L(X, s)$ of Calabi-Yau manifold X. Moreover there is a relation between coefficients of Dirichlet expansion of $L(X, s)$ and number of $\mathbb{F}_p$-rational points of X for any prime p. After that we define modular forms and associate L-function.

Finally, using the above tools we state what the modularity means in the case of Calabi–Yau threefolds. The definition is inspiring by Taniyama-Shimura conjecture, which predicted a modularity of elliptic curves. Taniyama-Shimura-Weil conjecture was obtained by A. Wiles and it implies the Fermat Last Theorem. Rigid Calabi-Yau threefolds are also modular although the question asked by B. Mazur and D. van Straten — which modular forms can be realized as a modular form of some rigid Calabi-Yau threefold, is still open.

In the further part we shall introduce double octic Calabi-Yau threefolds, which are a double cover of $\mathbb{P}^3$ branched along an octic surface. The study of such double coverings was initiated by C. H. Clemens. S. Cynk and his co-authors extended the class of branch locus and studied new aspects.

The final part of this chapter and present thesis consists of authors main results. By studying a symplectic involutions ϕ acting on double octic threefolds X we give a sufficient conditions for $\mathrm{Fix}(\phi)$ to not contain a curve of positive genus (Proposition 5.3.1). This proposition applied to suitably chosen examples from Meyer's list of arrangements produces new quotients varieties X/ϕ, which are rigid Calabi-Yau threefolds. As a consequence of that construction we find new rigid realizations of weight 4 cusp forms 96k4B1 and 96k4E1 (Theorem 5.3.3).

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CHAPTER 1

PRELIMINARIES

In this chapter we shall collect the information from the Hodge theory that we need in further parts of present thesis. We shall recall the Hodge decomposition for Kähler manifolds and basic properties of the Hodge diamond. Then we state Künneth's formula for Hodge numbers and description of Hodge numbers of blow-ups. We end this section by summarizing very briefly the results from deformation theory of complex manifolds.

1.0.1 The Hodge decomposition

The centrallll objects in Hodge theory are Kähler manifolds, which are differentiable manifolds endowed with a complex structure and a Riemannian metric satisfying a certain compatibility condition. We reefer to [Huy05] for a complete introduction to Kähler manifolds.

Let X be a complex manifold with the induced complex structure I.

Definition 1.0.1. A Riemannian metric g on X is an *hermitian structure* on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure I_x. The induced real $(1,1)$-form $\omega := g\,(I(\ ),(\ ))$ is called the *fundamental form.*

Definition 1.0.2. A Kähler structure (or Kähler metric) on X is an hermitian structure g for which the fundamental form ω is closed, i.e. $\mathrm{d}\omega = 0$. The complex manifold endowed with the Kähler structure is called a *Kähler manifold.*

Any projective variety is a Kähler manifold. The converse statement is also true under some assumptions:

Theorem 1.0.3 (Beauville)**.** *If X is a compact, Kähler variety such that $H^2(X, \mathcal{O}_X) = 0$, then X is projective.*

Let X be a complex, compact, projective variety of dimension $\dim X = n$. For any $0 \leq p, q \leq n$ we define vector spaces $H^{p,q}(X)$ with the following properties:

1. Vector spaces $H^{p,q}(X)$ are finite dimensional.
2. (Dolbeault theorem) There exists an isomorphism
$$H^{p,q}(X) \simeq H^q(X, \Omega_X^p).$$
3. (The Hodge decomposition)
$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$
4. (Serre duality) For any vector bundle E on X
$$H^p(X, E) \simeq H^{n-p}(X, E^\vee \otimes \omega_X)^\vee,$$
in particular
$$H^{p,q}(X) \simeq H^{n-p,n-q}(X)^\vee.$$
5. (The Hodge's symmetry)
$$H^{p,q}(X) = \overline{H^{q,p}(X)}.$$

Remark 1.0.4. The vector space $H^{p,q}(X)$ is called the *Dolbeault cohomology* of X and

$$H^{p,q}(X) = \frac{\ker\left(\overline{\partial}\colon \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(X)\right)}{\operatorname{im}\left(\overline{\partial}\colon \mathcal{A}^{p,q-1}(X) \to \mathcal{A}^{p,q}(X)\right)},$$

where $\mathcal{A}^{p,q}$ are sheaves of sections of complex vector bundles

$$\bigwedge^{p,q} X := \bigwedge^{p} \left(T^{1,0}X\right)^* \otimes_{\mathbb{C}} \bigwedge^{q} \left(T^{0,1}X\right)^*,$$

where

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X$$

is a direct-sum decomposition induced by a complex structure on X.

Since $H^{p,q}(X)$ is finite dimensional, the number $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$ is well defined and it is called the *Hodge number* of X. Hodge numbers are one of the most important invariants of a complex variety.

The Hodge numbers $h^{p,q}(X)$ may be visualized as the *Hodge diamond*, which has the following shape:

$$\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{1,0} & & h^{0,1} & & & \\
 & & & & \vdots & & & & \\
 & & \iddots & & & & \ddots & & \\
 & h^{n-1,0} & & & \vdots & & & h^{0,n-1} & \\
h^{n,0} & & h^{n-1,1} & \cdots & & \cdots & h^{1,n-1} & & h^{0,n} \\
 & h^{n,1} & & & \vdots & & & h^{1,n} & \\
 & & \ddots & & & & \iddots & & \\
 & & & & \vdots & & & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}$$

Serre duality shows that the Hodge diamond is invariant under central reflection. Moreover, by the Hodge symmetry the Hodge diamond is invariant under the reflections in diagonals.

The Hodge decomposition yields the following formula for Betti numbers

$$b_k(X) = \sum_{i+j=k} h^{i,j}(X),$$

hence k-th Betti number is the sum of numbers in k-th row of the Hodge diamond. Consequently the *Euler characteristic* of X may also be written in terms of Hodge numbers

$$\chi(X) := \sum_{k=0}^{2n}(-1)^k b_k(X) = \sum_{k=0}^{2n}(-1)^k \sum_{i+j=k} h^{i,j}(X). \tag{0-1}$$

1.0.2 The Künneth formula

In this section we recall Künneth's formula describing the de Rham cohomology of a product of complex manifolds, for a reference see the classical book [BT82] of R. Bott and L. W. Tu.

Theorem 1.0.5. *Let X and Y be a complex manifolds. Then for all $n \geq 0$ the following formula holds*

$$H^n(X \times Y, \mathbb{C}) = \bigoplus_{p+q=n} H^p(X, \mathbb{C}) \otimes H^q(Y, \mathbb{C}).$$

As the Hodge decomposition is compatible with Künneth's formula we get the following corollary:

Corollary 1.0.6. *If X and Y are complex manifolds, then*

$$h^{r,s}(X \times Y) = \sum_{\substack{r_1+r_2=r \\ s_1+s_2=s}} h^{r_1,s_1}(X) \cdot h^{r_2,s_2}(Y).$$

1.0.3 The Hodge numbers of blow-ups

Let X be a Kähler manifold, and $Y \subset X$ be a submanifold. The blown-up manifold $\mathrm{Bl}_Y(X)$ is still Kähler (see [Voi02], proposition 3.24). Let $E = \pi^{-1}(Y)$ be the exceptional divisor, where $\pi\colon \mathrm{Bl}_Y(X) \to X$ is a blow-up. Then E is a projective bundle of rank $r-1$, where $r = \operatorname{codim} Y$.

Theorem 1.0.7 ([Voi02], Theorem 7.31, p. 180)**.** *For any $k \geq 0$, there exists isomorphism*

$$H^k(\mathrm{Bl}_Y(X), \mathbb{C}) \simeq H^k(X, \mathbb{C}) \oplus \left(\bigoplus_{i=0}^{r-2} H^{k-2i-2}(Y, \mathbb{C}) \right).$$

To obtain the Hodge numbers of $\mathrm{Bl}_Y(X)$ it is enough to compare the above theorem with the Hodge decomposition.

1.0.4 General results from deformation theory

Deformation theory is the effect of fundamental work of K. Kodaira and D. C. Spencer. Their deformation techniques had received many deep applications in the Italian school of algebraic geometry. We shall briefly sketch the main definitions and statements in this theory. We refer to [Har10, GHJ03, Kod86] for a solid introduction to deformation theory.

Let X be a compact complex manifold. A *deformation* of X consists of a smooth proper morphism $\mathcal{X} \to S$, where $\mathcal{X}$ and S are connected complex spaces and $X \simeq \mathcal{X}_0$, where $0 \in S$ is a distinguished point (we consider only a germ $(S, 0)$). An *infinitesimal deformation* of X is a deformation with the base space $S = \mathrm{Spec}(\mathbb{C}[\varepsilon])$.

Theorem 1.0.8 ([GHJ03], Proposition 22.1)**.** *The isomorphism classes of infinitesimal deformations of a compact complex manifold X are parametrized by elements in $H^1(X, \mathcal{T}_X)$.*

Definition 1.0.9. A deformation $\mathcal{X} \to (S, 0)$ of X is called *universal* if for any other deformation $\mathcal{X}' \to (S', 0')$ there exists an unique morphism $\phi\colon S \to S'$ with $\phi(0) = 0'$, such that $\mathcal{X}' \simeq \phi^*(\mathcal{X})$.

The universal family is unique up to an isomorphism, and it is denoted by $\mathcal{X} \to \mathrm{Def}(X)$; space $\mathrm{Def}(X)$ is usually called the *Kuranishi space.*

Theorem 1.0.10 (Kuranishi)**.** *Let X be a compact, complex manifold with $H^0(X, \mathcal{T}_X) = 0$, then a universal deformation of X exists and it is universal for any of its fibers.*

Theorem 1.0.11 (Kodaira, Spencer)**.** *Tangent vectors to* $(\mathrm{Def}(X), 0)$ *at* 0 *are naturally identified with classes in $H^1(X, \mathcal{T}_X) = 0$. If also $H^2(X, \mathcal{T}) = 0$ then* $\mathrm{Def}(X)$ *can be identified with an open neighbourhood of* 0 *in $H^1(X, \mathcal{T}_X) = 0$, in particular* $\mathrm{Def}(X)$ *is smooth.*

Definition 1.0.12. Let X be a compact complex manifold that admits a universal deformation $\mathcal{X} \to \mathrm{Def}(X)$. We say that the deformations of X are *unobstructed* if $\dim \mathcal{T}_0 \mathrm{Def}(X) = \dim \mathrm{Def}(X)$.

CHAPTER 2

CALABI-YAU MANIFOLDS

Calabi-Yau threefolds are subject of intensive studies, motivated by their possible applications to physics in the so called *string theory.* There are several inequivalent definitions of Calabi-Yau varieties (see chapter 1 of [GHJ03]). According to Yau's proof of Calabi conjecture in [Yau77], it can be said that all definitions are "almost" equivalent. Calabi-Yau manifolds have been studied also due to their appearance in the famous Beauville-Bogomolov decomposition theorem.

String theory predicts that our universe is not just four-dimensional (time-space), but ten-dimensional. The extra six dimensions or complex three "compactify" to a Calabi-Yau variety. The most famous unsolved mathematical problem inspired by string theory is the *Mirror Symmetry Conjecture.* String theorists observed that if "strings" were to exist, then they must come in pairs. Consequently, for any non-rigid Calabi-Yau threefold X there should exists a "mirror partner" Y of X. Mathematicians reformulate this phenomenon into symmetries between Hodge diamonds of X and Y.

Calabi-Yau manifolds can be considered as the three-dimensional counterpart of elliptic curves, thus apart from the motivations coming from physics, Calabi-Yau manifolds provide a very interesting framework to study from the point of view of classification of algebraic threefolds and arithmetic. After proving Taniyama-Shimura-Weil conjecture by A. Wiles in [Wil95], mathematicians started to study a correspondences between Calabi-Yau threefolds and certain modular forms.

$K3$ surfaces which are two-dimensional Calabi-Yau manifolds provides beautiful and rich theory. The study and classification of non-symplectic automorphisms pioneered by V. V. Nikulin in [Nik79a] is and important subject of research.

In the present chapter we define Calabi-Yau manifolds and summarize classical results concerning these manifolds. Main part of this chapter will be devoted to study $K3$ surfaces and particular kinds of automorphisms acting on them.

2.1 General information

In this section we define a Calabi-Yau manifold and describe its Hodge diamond in low dimensions. Also, we formulate simplest version of the famous mirror symmetry conjecture. There are many inequivalent definitions of Calabi-Yau manifolds, for a reference see [Joy00].

Definition 2.1.1. A Calabi-Yau manifold X is a complex compact Kähler d-fold X satisfying

(i) $K_X = \mathcal{O}_X$,

(ii) $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < d$.

Remark 2.1.2. The above definition is equivalent to the following conditions

(i) there are no global holomorphic i-forms on X,

(ii) there exists a nowhere vanishing holomorphic d-form on the manifold X.

Remark 2.1.3. For a Calabi-Yau manifold X of dimension d,

$$h^{i,0}(X) = h^{0,i}(X) = 0, \quad \text{for} \quad 0 < i < d$$

and

$$h^{0,0}(X) = h^{0,d}(X) = h^{d,0}(X) = h^{d,d}(X) = 1.$$

Remark 2.1.4. A non-trivial generator ω_X of $H^{d,0}(X) \simeq \mathbb{C}$ is called a *period* of X. For any automorphism $\alpha_X \in \mathrm{Aut}(X)$, the induced mapping α_X^* acts on $H^{d,0}(X)$ and $\alpha_X^*(\omega_X) = \lambda_\alpha \omega_X$, for some $\lambda_\alpha \in \mathbb{C}^*$. We shall analyse a period of Calabi-Yau manifold in the case of dimension two.

Calabi-Yau manifolds are "building blocks" of compact Kähler Ricci flat manifold, which follows from the famous Beauville-Bogomolov decomposition theorem:

Theorem 2.1.5 (Beauville-Bogomolov, [Bea83, Bog74])**.** *Let X be a compact Kähler Ricci flat manifold. Then X has a finite unramified cover Y with*

$$Y \simeq \mathbb{T} \times \prod_i H_i \times \prod_i C_i,$$

where

(i) $\mathbb{T}$ *is a complex torus i.e.* $\mathbb{C}^n/\Lambda$, *where* $\Lambda \simeq \mathbb{Z}^{2n}$ *is a lattice in* $\mathbb{C}^n$.

(ii) *Each H_i is a Hyperkähler manifold i.e. simply connected holomorphic symplectic manifold with* $\dim H^2(H_i, \mathcal{O}_{H_i}) = 0$.

(iii) *Each C_i is a simply connected Calabi-Yau manifold.*

By Yau's proof of Calabi's conjecture (see [Yau77]) having a Ricci flat metric is equivalent to having trivial first Chern class.

Let us recall the fundamental theorem due to V. Batyrev in [Bat99]:

Theorem 2.1.6. *For any birational n-dimensional Calabi-Yau manifolds X and Y the Hodge numbers are equal.*

Examples 2.1.7.

1. A Calabi-Yau manifold of dimension one is an elliptic curve with Hodge diamond:

$$\begin{matrix} & 1 & \\ 1 & & 1 \\ & 1 & \end{matrix}$$

2. A two dimensional Calabi-Yau manifold is called a $K3$ *surface*, with the following Hodge diamond:

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

3. A three dimensional Calabi-Yau manifold X is called *Calabi-Yau threefold.* From 4 and 5 it follows that the Hodge diamond of X has the following shape:

$$\begin{matrix} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{matrix}$$

Moreover from (0-1) we obtain a brief formula for Euler characteristic of X i.e.

$$\chi(X) = 2\left(h^{1,1}(X) - h^{2,1}(X)\right).$$

Conjecturally, the number $|\chi(X)|$ should be bounded, and currently the maximal known is 960 (it is achieved by a hypersurface in the weighted projective space $\mathbb{P}(1, 1, 12, 28, 42)$). However there is also conjecture that there are infinity many families of Calabi-Yau threefolds.

Note that for a Calabi-Yau threefold X:

$$h^{1,1}(X) = \text{rank Pic}(X) \quad \text{and} \quad h^{2,1}(X) = \dim \text{Def}(X).$$

Theorem 2.1.8 ([Ran92])**.** *If X is a Calabi-Yau manifolds, then deformations of X are unobstructed.*

Theorem 2.1.9 (Bogomolov, Todorov)**.** *For n-dimensional Calabi-Yau manifold the Kuranishi space* $\text{Def}(X)$ *exists and*

$$\dim \text{Def}(X) = h^{n-1,1}(X).$$

Bogomolov-Todorov theorem gives raise to the following definition:

Definition 2.1.10. A Calabi-Yau manifold X is called *rigid* if $h^1\mathcal{T}_X = 0$.

Physicists have discovered a phenomenon for Calabi-Yau threefolds, which is called a *Mirror Symmetry.* In the simplest version, it predicts that for a given non-rigid Calabi-Yau threefold X there exists a Calabi-Yau threefold Y such that

$$h^{1,1}(X) = h^{2,1}(Y) \quad \text{and} \quad h^{2,1}(X) = h^{1,1}(Y).$$

In particular it implies that $\chi(X) = -\chi(Y)$.

There are many examples of Calabi-Yau threefolds satisfying mirror symmetry conjecture, for example Borcea-Voisin Calabi-Yau 3-folds, which will be defined in the further part of present thesis. The largest amount of examples comes from the construction given by V. Batyrev in [Bat94]. He started with a reflexive polytope in dimension 4, by taking corresponding toric 4-fold, he considered a generic anti-canonical section and constructed in this way a Calabi-Yau orbifold. After that, he found a smooth desingularization of resulting variety, and obtained Calabi-Yau variety.

It is worth to mention that there is more complicated (surprising) reformulation of mirror symmetry due to M. Kontschevich, which involves notion of derived and Fukaia categories of variety. That version of mirror symmetry were intensive studied in recent years; for a good reference in this topic, see [Huy06].

2.2 $K3$ surfaces

Two dimensional Calabi-Yau manifolds are called $K3$ *surfaces.* André Weil named them in honour of three algebraic geometers: Kummer, Kähler and Kodaira, and the mountain K2 in Kashmir.

Note that definition 2.1.1 tells nothing about the fundamental group. It can be shown that any $K3$ surface is deformation equivalent to a quartic hypersurface in $\mathbb{P}^3$, in particular all $K3$ surfaces are diffeomorphic and simply-connected (see [BHPVdV04]).

Examples 2.2.1.

1. Let X be a smooth quartic surface in $\mathbb{P}^3$, then $\omega_X \simeq \mathcal{O}_X$ by the adjunction formula (see [Har77], chapter 5, proposition 1.5). Moreover, from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0,$$

by taking cohomology, we get that $h^1(X, \mathcal{O}_X) = 0$.

2. More generally, a smooth complete intersection of type $(d_1, \ldots, d_n)$ in $\mathbb{P}^{n+2}$ is a $K3$ surface iff $\sum_{i=1}^{n} d_i = n + 3$. Assuming that the complete intersection is non-degenerate i.e. $d_i \geq 2$, we get three possibilities:

$n = 1$	$d_1 = 4$	$\mathbb{P}^3$
$n = 2$	$(d_1, d_2) = (2, 3)$	$\mathbb{P}^4$
$n = 3$	$(d_1, d_2, d_3) = (2, 2, 2)$	$\mathbb{P}^5$

3. Consider a double cover $\pi \colon X \to \mathbb{P}^2$ branched along a smooth curve S of degree 6. Then $\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$, hence:

$$H^1(X, \mathcal{O}_X) = H^1(\mathbb{P}^2, \pi^* \mathcal{O}_X) = H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \oplus H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0.$$

Canonical bundle formula for branched coverings (see [BHPVdV04]) implies that:

$$\omega_X \simeq \pi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}(3)) \simeq \mathcal{O}_X,$$

thus X is a $K3$ surface.

4. Let A be an Abelian surface and let ι be the involution of A given by $\iota\colon A \ni a \to -a \in A$. The fixed locus of ι consists of 16 two-torsion points. Let $\pi\colon \widetilde{A} \to A$ be the blow-up of these 16 points. The involution ι extends to an involution τ of $\widetilde{A}$; denote by X the quotient $X = \widetilde{A}/\langle\tau\rangle$. Then X is a $K3$ surface, called the *Kummer surface* (for a details see [Bea96]).

Remark 2.2.2. U. Persson in [Per85] investigated similar construction to (3) with branch locus consisting of a sum of six lines. As a result we get a $K3$ surface with large Picard number, in particular a singular $K3$ surface ($\rho = 20$). In the last chapter of this thesis we consider double cover of $\mathbb{P}^3$ branched along eight planes, which are Calabi-Yau threefolds.

2.3 Automorphisms of $K3$ surfaces

An automorphism $\alpha_S \in \mathrm{Aut}(S)$ of a $K3$ surface S is called *symplectic* if it preserves a period of S i.e. $\alpha_S^*(\omega_S) = \omega_S$. If α_S does not preserve a period then it is called *non-symplectic.* If additionally α_S is of finite order n and $\alpha_S^*(\omega_S) = \zeta_n\omega_S$, where ζ_n is a primitive n-th root of unity, then it is called *purely non-symplectic.*

Remark 2.3.1. If α_S is a non-symplectic automorphism of order n, then its action on a period is given by an n-th root of unity. Moreover α_S is purely non-symplectic if its every power is non-symplectic.

Lemma 2.3.2. *Let X be a complex m-dimensional manifold and let α_X be an automorphism of X of order n. Then for any fixed point x of α_X there exists a local holomorphic coordinate system $(z_1, z_2, \ldots, z_m)$ around x and n-th roots of unity $\lambda_1, \lambda_2, \ldots, \lambda_m$ such that $\alpha_X(z_1, z_2, \ldots, z_m) = (\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_m z_m)$ and $\alpha_X^*(\omega_X) = \lambda_1\lambda_2 \cdot \ldots \cdot \lambda_m\omega_X$.*

Proof. We can assume that $x = 0$ and consider neighbourhood $0 \in U \subset \mathbb{C}^m$. Following

[Car57], for $y \in U$ define:

$$f(y) := \frac{1}{n}\sum_{i=1}^{n}(\mathrm{d}_0\alpha_X)^{-i} \circ \alpha_X^i(y),$$

then $\mathrm{d}_0 f = \frac{1}{n}\sum_{i=1}^{n}(\mathrm{d}_0\alpha_X)^i \cdot (\mathrm{d}_0\alpha_X)^{-i} \equiv \mathrm{id}$, hence $(z_1, z_2, \ldots, z_m) := f(y)$ is a local coordinate system around 0.

Note that:

$$f \circ \alpha_X(y) = \frac{1}{n}\sum_{i=1}^{n}\alpha_X^{i+1}(y) \cdot (\mathrm{d}_0\alpha_X)^{-i} = \mathrm{d}_0\alpha_X \cdot f(y),$$

thus with respect to a new coordinates system $(z_1, z_2, \ldots, z_n)$ we can assume that α_X is linear. Since α_X is of finite order, after linear change of coordinates it is represented by a matrix with roots of unity on diagonal. □

Corollary 2.3.3. *Let α_S be a purely non-symplectic automorphism of order n of a $K3$ surface S. Then the action of α_S may be locally linearized and diagonalized at a fixed point $p \in \mathrm{Fix}(\alpha_S)$, so the possible local actions are*

$$\begin{pmatrix} \zeta_n^{t+1} & 0 \\ 0 & \zeta_n^{n-t} \end{pmatrix}, \quad \text{for } t = 0, 1, \ldots, n-1.$$

Clearly, if $t = 0$, then p belongs to a smooth curve fixed by α_S, otherwise p is an isolated point.

Theorem 2.3.4 ([AST11], Lemma 2.2, p. 5)**.** *Let S be a $K3$ surface and let α_S be a non-symplectic automorphism of S of order n. Then there are three possibilities*

1. $\mathrm{Fix}(\alpha_S) = \varnothing$; *in this case $n = 2$,*

2. $\mathrm{Fix}(\alpha_S) = E_1 \cup E_2$, *where E_1, E_2 are disjoint smooth elliptic curves; in this case $n = 2$,*

3. $\mathrm{Fix}(\alpha_S) = C \cup R_1 \cup R_2 \cup \ldots R_{k-1} \cup \{p_1, p_2, \ldots, p_h\}$, *where p_i are isolated fixed points, R_i are smooth rational curves and C is the curve with highest genus $g(C)$.*

Proof. From the corollary 2.3.3 it follows that the fixed locus of α_S consists of disjoint curves and isolated points. Denote fixed curves by $R_0, R_1, \ldots, R_{k-1}$.

Suppose that $g(R_0) > 1$. The adjunction formula yields $R_i^2 = 2g(R_i) - 2$, hence $R_0^2 > 0$. Since curves R_i and R_j are disjoint, then $R_i R_j = 0$ for $i \neq j$, thus the Hodge index formula implies that $2g(R_i) - 2 = R_i^2 \leq 0$ for any $1 \leq i \leq k-1$. However R_i is an effective divisor, so $g(R_i) = 0$. Therefore at most one curve has genus strictly bigger than 1, and if there is such a curve, then all remaining curves are rational.

Suppose that $g(R_0) = 1$. Then other curves may be elliptic or rational. Suppose that R_0 and R_1 are elliptic. They give raise to elliptic fibration $\pi\colon S \to \mathbb{P}^1$ which is α_S-invariant (see [ST17]). Local action of α_S on R_0 is given by $\alpha_S^*(x,y) = (x, \zeta_n y)$, hence α_S descends to an automorphism of $\mathbb{P}^1$. The automorphism of $\mathbb{P}^1$ has two fixed points, which implies that $\mathrm{Fix}(\alpha_S)$ consists of R_0 and R_1. □

At this moment a natural problem arrives: classify all non-symplectic automorphisms (of a $K3$ surface) of given order. In the case of elliptic curve E, by the well known result (see [Sil09]) the automorphism group of E has order dividing 6. For $K3$ surfaces the problem is much more difficult.

Theorem 2.3.5 (Nikulin, [Nik79a])**.** *Let S be a $K3$ surface. If α_S is a non-symplectic automorphism of S of finite order n, then $\phi(n) \leq 21$, where ϕ is the Euler ϕ-function. In particular, $n \leq 66$ and when α_S has prime order, its order is at most* 19.

Therefore the group of purely non-symplectic automorphisms is finite. All the orders of such groups have been determined in [MO98] by N. Machida and K. Oguiso. A non-symplectic automorphisms of prime order have been classified by M. Artebani, A. Sarti, S. Taki in [AST11]. Some authors have started to investigate non-symplectic automorphisms of composite order: D. Al Tabbaa and A. Sarti in [TS17] (order 8), J. Dillies in [Dil12b] (order 6), A. Sarti and A. Garbagnati in [GS13] (order $2p$, where p

is a prime) and J. Keum in [Keu15] (order 66). Also, S. Brandhorst in [Bra17] pursued the question when an automorphism determines a $K3$ surface up to an isomorphism.

CHAPTER 3

QUOTIENTS OF PROJECTIVE VARIETIES

Quotients of projective varieties by a finite group action provide many examples of Calabi-Yau manifolds. In most cases quotients are singular and we have to perform a resolution of singularities as a second step of the construction. If we want to get a Calabi-Yau manifold as a result we have to consider a *crepant resolution* of singularities. The term "crepant" was coined by M. Reid by removing the prefix "dis" from the word "discrepant", to indicate that the resolutions have no discrepancy (see [CR00]) in the canonical class. Unfortunately, such a resolution exists only for very special types of singularities. Moreover in those cases it is difficult to construct them.

In this chapter we shall define crepant resolutions and state basic results ensuring existence of such resolution for certain types of singular varieties. Also, we shall recall a definition of Chen-Ruan cohomology and orbifold (stringy) formula.

3.1 Crepant resolution

Let G be a finite group acting on a projective complex variety X. In points corresponding to a points of X with non-trivial stabilizer, the projective variety X/G may be singular. To avoid this problem we need to find a *resolution of singularities* of X/G.

Definition 3.1.1. Pair $(\widetilde{X}, \pi)$, where $\pi\colon \widetilde{X} \to X$ is a proper, regular and birational map and $\widetilde{X}$ is a smooth algebraic variety, is called a resolution of singularities of X if

$$\pi\big|_{\pi^{-1}(\mathrm{Reg}(X))}\colon \pi^{-1}\left(\mathrm{Reg}(X)\right) \to \mathrm{Reg}(X)$$

is an isomorphism.

By Hironaka's desingularization theorem, any complex algebraic variety X has a desingularization. As we want to control the canonical divisor, we shall now restrict considerations to the case, where X admits canonical line bundle i.e. has Gorenstein singularities (for a details see [Rei87]).

Definition 3.1.2. Let X be a complex algebraic variety with Gorenstein singularities, and let $(\widetilde{X}, \pi)$ be a resolution of X. We say that that $\widetilde{X}$ is a *crepant resolution* of X if

$$\pi^*(K_X) = K_{\widetilde{X}}.$$

Because singularities of a complex orbifold X are locally isomorphic to a quotient variety $\mathbb{C}^n/G$ for a finite $G \subseteq \mathrm{Gl}_n(\mathbb{C})$, any crepant resolution of X are locally isomorphic to a crepant resolution of $\mathbb{C}^n/G$. Thus the study of existence of a crepant resolution of variety $\mathbb{C}^n/G$ is really important.

In dimension 2, a crepant resolution of a quotient by a subgroup of $\mathrm{Sl}_2(\mathbb{C})$ always exists and is unique as follows from a classical result of Klein.

Theorem 3.1.3. *Let G be any finite subgroup of* $\mathrm{Sl}_2(\mathbb{C})$. *Then surface $\mathbb{C}^2/G$ admits an unique crepant resolution.*

Surfaces $\mathbb{C}^2/G$ are called *Kleinian singularities* or *Du Val surface singularities.* There is 1-1 correspondence between non-trivial finite subgroups $G \subset \mathrm{Sl}_2(\mathbb{C})$ and *the Dynkin diagrams* of type $A_k (k \geq 1)$, $D_k (k \geq 4)$, E_6, E_7 and E_8.

The correspondence between Kleinian singularities $\mathbb{C}^2/G$, Dynkin diagrams and other areas of mathematics is known as the *McKay correspondence.*

In dimension 3 situation is a little bit more subtle. A crepant resolution always exist but it is not unique. Any two crepant resolutions are related by a sequence of the so called *flops* (for a details see [Kol89]).

Theorem 3.1.4 (Roan, [Roa96])**.** *Let G be a finite subgroup of* $\mathrm{Sl}_3(\mathbb{C})$*. Then* $\mathbb{C}^3/G$ *admits a crepant resolution.*

Roan's proof is by explicit construction, using classification of finite subgroups of $\mathrm{Sl}_3(\mathbb{Z})$. If $n \geq 4$, then singularities are less well understood and a crepant resolution exists only in rather special cases.

Example 3.1.5. For a subgroup $\{-1,+1\} \subseteq \mathrm{Sl}_4(\mathbb{C})$, variety $\mathbb{C}^4/\{-1,+1\}$ does not admit any crepant resolution! This follows from the results of Reid, Shephard-Barron and Tai (see [MS84]).

Let us recall a definition of an age of a matrix, which is important also in the further context of this paper.

Definition 3.1.6. For $G \in \mathrm{Gl}_n(\mathbb{C})$ of order m, let $e^{2\pi i a_1}, e^{2\pi i a_2}, \ldots, e^{2\pi i a_n}$ be eigenvalues of G for some $a_1, a_2, \ldots, a_n \in [0,1) \cap \mathbb{Q}$. The value of the sum $a_1 + a_2 + \ldots + a_n$ is called the *age* of G and is denoted by $\mathrm{age}(G)$.

Remark 3.1.7. The age of G is an integer if and only if $\det G = 1$ i.e. $G \in \mathrm{Sl}_n(\mathbb{C})$.

We and this section with a conjecture, which is crucial in study of a crepant resolutions.

Conjucture 3.1.8 (McKay correspondence)**.** Let G be a finite subgroup of $\mathrm{Sl}_n(\mathbb{C})$ and $(\widetilde{X}, \pi)$ a crepant resolution of $X := \mathbb{C}^n/G$. Then there exists basis of $H^*(X,\mathbb{Q})$ consisting of algebraic cycles in 1-1 correspondence with conjugacy classes of G, such that conjugacy classes with age k correspond to basis elements of $H^{2k}(X,\mathbb{Q})$. In particular b_{2k} is the number of conjugacy classes G with age k, and $b_{2k+1} = 0$, so the Euler characteristic $\chi(X) = \#\mathrm{Conj}(G)$ — the number of conjugacy classes of G.

3.2 Orbifold's cohomology

In [CR04] W. Chen and Y. Ruan introduced a cohomology theory for orbifolds. We consider varieties X/G, where X is a projective variety and G is a finite group acting on X viewed as orbifold.

Definition 3.2.1. For a variety X/G define the Chen-Ruan cohomology by

$$H^{i,j}_{\mathrm{orb}}(X/G) := \bigoplus_{[g]\in \mathrm{Conj}(G)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{\mathrm{C}(g)},$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G (we choose a representative g of each conjugacy class), $\mathrm{C}(g)$ is the centralizer of g, $\Lambda(g)$ denotes the set of irreducible connected components of the set fixed by $g \in G$ and $\mathrm{age}(g)$ is the age of the matrix of linearized action of g near a point of U.

The dimension of $H^{i,j}_{\mathrm{orb}}(X/G)$ will be denoted by $h^{i,j}_{\mathrm{orb}}(X/G)$.

Remark 3.2.2. If the group G is cyclic of a prime order p, then we can pick a generator α and the above formula simplifies to

$$H^{i,j}_{\mathrm{orb}}(X/G) = H^{i,j}(X)^G \oplus \bigoplus_{U\in\Lambda(\alpha)} \bigoplus_{k=1}^{p-1} H^{i-\mathrm{age}(\alpha^k),\, j-\mathrm{age}(\alpha^k)}(U).$$

We have the following theorem:

Theorem 3.2.3 ([Yas04], Theorem 1.1, p. 2)**.** *Let G be a finite group acting on an algebraic smooth variety X. If there exists a crepant resolution $\widetilde{X/G}$ of variety X/G, then the following equality holds*

$$h^{i,j}(\widetilde{X/G}) = h^{i,j}_{\mathrm{orb}}(X/G).$$

Example 3.2.4. Let E be a complex elliptic curve and take

$$E_0 := \{(x_1, x_2, \ldots, x_{n+1}) \in E^{n+1} |\, x_1 + x_2 + \ldots + x_{n+1} = 0\} \simeq E^n.$$

Alternating group A_{n+1} acts on E^{n+1} and fixes E_0. Consider

$$X_n := E_0/A_{n+1} \cong E^n/A_{n+1}.$$

In [PR05] K. Paranjape and D. Ramakrishnan proved the following theorem:

Theorem 3.2.5. *For $n \leq 3$, variety X_n admits a crepant resolution $\widetilde{X_n}$, which is Calabi-Yau.*

We will compute the Hodge numbers of $\widetilde{X_2}$ and $\widetilde{X_3}$ using the orbifold cohomology formula.

If $n = 2$, then $A_3 \simeq C_3 = \langle \sigma \rangle$, where

$$\sigma \colon E^2 \ni (x, y) \to (-x - y, x) \in E^2.$$

We see that

$$\#\Lambda(\sigma) = \#\Lambda(\sigma^2) = \#E[3] = 9$$

and locally the action of σ may be linearized to a matrix

$$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix},$$

hence $\operatorname{age}(\sigma) = \operatorname{age}(\sigma^2) = 1$. From the orbifold formula we get

$$H^{i,j}_{\mathrm{orb}}(E^2/A_3) = H^{i,j}(E \times E)^{A_3} \oplus \bigoplus_{P \in \Lambda(\sigma)} \bigoplus_{k=1}^{2} H^{i-\operatorname{age}(\sigma^k),\, j-\operatorname{age}(\sigma^k)}(P),$$

thus $h^{1,1}(\widetilde{X_2}) = 18 + h^{1,1}(E \times E)^{A_3}$.

Action of σ on the form

$$A\,\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + B\,\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} + C\,\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1} + D\,\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2},$$

where $A, B, C, D \in \mathbb{C}$, leads to the following system of equations

$$\begin{cases} A - B + C - D = A \\ A - D = B \\ A = C \\ A - B = D, \end{cases}$$

hence $C = A$ and $D = A - B$, so $h^{1,1}(E \times E)^{A_3} = 2$ and $h^{1,1}(\widetilde{X_2}) = 20$, which implies that $\widetilde{X_2}$ is a $K3$ surface.

A three dimensional case is slightly more complicated. Again the orbifold formula yields

$$H^{i,j}_{\mathrm{orb}}(E^3/A_4) = \bigoplus_{[g]\in \mathrm{Conj}(A_4)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{\mathrm{C}(g)}. \tag{2-1}$$

Now we need to analyze the action of a representative of all conjugacy classes of A_4 f.i. id, $(1,2,3)$, $(1,3,2)$, $(1,2)(3,4)$.

Action of id. Consider an action of A_4 on the form

$$\sum_{1\le i,j\le 3} A_{3i+j-3}\, \mathrm{d}z_i \wedge \mathrm{d}\overline{z_j},$$

where $A_1, A_2, \ldots, A_9 \in \mathbb{C}$. For $(1,2,3)$ and $(1,3,4)$ we get equations

$$\begin{cases} A_2 = A_3 = A_4 = A_6 = A_7 = A_8 := B \\ A_1 = A_5 = A_9 := A \\ A = 2B. \end{cases}$$

Moreover one can check that the form:

$$2\,\mathrm{d}z_1 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_2} + \mathrm{d}z_1 \wedge \mathrm{d}\overline{z_3} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_1} + 2\,\mathrm{d}z_2 \wedge \mathrm{d}\overline{z_2} + \mathrm{d}z_2 \wedge \mathrm{d}\overline{z_3} +$$
$$+ \mathrm{d}z_3 \wedge \mathrm{d}\overline{z_1} + \mathrm{d}z_3 \wedge \mathrm{d}\overline{z_2} + 2\,\mathrm{d}z_3 \wedge \mathrm{d}\overline{z_1}$$

is invariant under the action by A_4, hence $h^{1,1}(E \times E \times E)^{A_4} = 1$. Similar approach leads to the equality $h^{2,1}(E \times E \times E)^{A_4} = 1$.

Action of $g = (1,2,3), (1,3,2)$. The action of $g = (1,2,3), (1,3,2)$ on E^3 is given by

$$\widetilde{g}\colon E \times E \times E \ni (x,y,z) \to (y,z,x) \in E \times E \times E$$

and

$$\widetilde{g}\colon E \times E \times E \ni (x,y,z) \to (z,x,y) \in E \times E \times E,$$

respectively. Hence actions are represented by matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}.$$

In both cases $\operatorname{age}(g) = 1$, $\Lambda(g) = \{(x, x, x)\}_{x \in E} \simeq E$ and $\Lambda(g)$ is invariant under the action of centralizer of g, which is equal to $\langle(1, 2, 3)\rangle$.

Action of $g = (1, 2)(3, 4)$. The action of g is given by

$$\widetilde{g}\colon E \times E \times E \ni (x, y, z) \to (y, x, -x - y - z) \in E \times E \times E,$$

with a representative matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

thus

$$\Lambda(g) = \bigcup_{\tau \in E[2]} \{(x, x, \tau - x)\}_{x \in E} \quad \text{and} \quad \operatorname{age}(g) = 1.$$

The action of $(1, 3)(2, 4)$ and $(1, 4)(2, 3)$ on curves $\{(x, x, \tau - x)\}_{x \in E} \simeq E$ leads to curves $\{(\tau - x, \tau - x, x)\}_{x \in E}$. Note that

$$\mathrm{C}(g) = \{(1, 2)(3, 4), (1, 3)(2, 4), (1, 4)(2, 3)\}.$$

Since $\phi((x, x, \tau - x)) = (\tau - x, \tau - x, x)$, where

$$\phi\colon E^3 \ni (x, y, z) \to (\tau - x, \tau - y, \tau - z) \in E^3$$

is an involution, we see that $U/\mathrm{C}(g) \simeq E/(\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{P}^1$ for any $U \in \Lambda(g)$.

Finally, from (2-1) we deduce $h^{1,1}(\widetilde{X_3}) = 7$ and $h^{2,1}(\widetilde{X_3}) = 3$.

Remark 3.2.6. The quotients of a product of three elliptic curves by a finite group were studied by M. Donten-Bury in her MSc thesis (see [Don11]). She classified a finite index subgroups of $\mathrm{Sl}_3(\mathbb{Z})$ and found Hodge diamonds of resulting quotients, using different methods.

3.3 Orbifold Euler characteristic

Let G be a finite group acting on a compact differentiable manifold. Then the Euler characteristic of the quotient X/G may be computed using Lefschetz fixed point formula

$$e(X/G) = \frac{1}{\#G} \sum_{g \in G} e(X^g),$$

where X^g denotes the set of points $x \in X$ such that $gx = x$.

If we consider the quotient X/G as an orbifold, then motivated by string theory L. Dixon, J. Harvey, C. Vafa and E. Witten in [DHVW85, DHVW86] introduced *orbifold (stringy) Euler characteristic*, by the following formula:

$$e_{\text{orb}}(X/G) := \frac{1}{\#G} \sum_{\substack{(g,h) \in G \times G \\ gh = hg}} e(X^g \cap X^h).$$

In the case of algebraic variety X the following theorem holds

Theorem 3.3.1 ([Roa89], Theorem 2, p. 534)**.** *Let G be a finite abelian group acting on smooth algebraic variety X. If there exists a crepant resolution $\widetilde{X/G}$ of variety X/G, then the following equality holds*

$$e(\widetilde{X/G}) = e_{\text{orb}}(X/G).$$

If the group G is abelian, then under some assumptions this theorem has been proved by S. S. Roan [Roa89]. Some particular examples of a group G were studied by T. Höfer and F. Hirzebruch in [HH90].

CHAPTER 4

CALABI-YAU MANIFOLDS OF BORCEA-VOISIN TYPE

One of the first important achievements in the theory of Calabi-Yau threefolds and in particular Mirror Symmetry, was the construction given independently by C. Borcea ([Bor97]) and C. Voisin ([Voi93]). Borcea and Voisin constructed families of Calabi-Yau threefolds using a non-symplectic involutions of $K3$ surfaces and elliptic curves. Moreover C. Voisin gave a construction of explicit mirror maps while C. Borcea found a conditions when these threefolds have complex multiplication. Attempts to generalize the Borcea-Voisin construction motivated studies of the non-symplectic automorphisms on $K3$ surfaces.

The Borcea-Voisin construction is actually similar to the one given by C. Vafa and E. Witten in [VW95]. They divided a product of three tori by a group of automorphisms preserving the volume form. This approach gave rise to abstract physical models studied by L. Dixon, J. Harvey, C. Vafa, E. Witten in [DHVW85, DHVW86]. Similar constructions i.e. quotients of products of tori by a finite group were classified by J. Dillies, R. Donagi, A. E. Faraggi an K. Wendland in [Dil07, DF04, DW09].

There are many generalizations of the above constructions. The first idea is to allow automorphisms of higher order. In [Roh10] Rohde constructed Calabi-Yau threefolds by taking a quotient of a product of an elliptic curve and a $K3$ surface by an automorphism of order 3, fixing only points or rational curves on the $K3$ surface. A. Molnar in his PhD thesis ([Mol15]) found another groups acting on a product of three elliptic curves and studied modularity of the resulting quotients. S. Cynk and K. Hulek study examples of threefolds (and higher dimensional varieties) using involutions and higher order automorphisms (see [CH07]); they also proved their modularity. Finally [CG16] A. Cattaneo and A. Garbagnati used purely non-symplectic automorphisms of order 3, 4 and 6 to generalized the Borcea-Voisin construction.

Another possibility is to take a quotient of a product of two $K3$ surfaces by a finite group. Such fourfolds were studied by J. Dillies in [Dil12a]. F. Reidegald divided $S \times \mathbb{P}^1$, where S is a $K3$ surface, by a cyclic group of order 3. He also found a desingularization of such quotients (see [Rei15]).

In the present chapter we briefly recall the original Borcea-Voisin construction. Then, we shall reproof formulas for the Hodge numbers of Calabi-Yau threefolds constructed by A. Cattaneo and A. Garbagnati, using the orbifold cohomology formula and the stringy Euler characteristic.

4.1 The classical Borcea-Voisin construction

Theorem 4.1.1 (Borcea-Voisin, [Bor97, Voi93])**.** *Let E be an elliptic curve and α_E an involution which does not preserve a period of E. Let S be a $K3$ surface with a non-symplectic involution α_S. Then a crepant resolution $\widetilde{X}$ of the quotient $X := (E \times S)/(\alpha_E \times \alpha_S)$ is a Calabi-Yau manifold with the Hodge numbers:*

$$h^{1,1}(\widetilde{X}) = h^{2,2}(\widetilde{X}) = 11 + 5N' - N \quad \text{and} \quad h^{2,1}(\widetilde{X}) = h^{1,2}(\widetilde{X}) = 11 + 5N - N',$$

where N is the number of curves in $\mathrm{Fix}(\alpha_S)$ and N' denotes the sum of genera of all curves in fixed locus.

From Nikulin's classification it follows that for any $K3$ surface S with non-symplectic involution α_S which fixes N curves with sum of genera equal to N', there exists a complementary surface S' and its non-symplectic involution α'_S with N' fixed curves with sum of genera equal to N. Thus we have the following corollary:

Corollary 4.1.2. *The pair* $\left(\widetilde{(E \times S)/(\alpha_E \times \alpha_S)},\ \widetilde{(E \times S')/(\alpha_E \times \alpha_{S'})}\right)$ *is a mirror pair.*

4.2 The Cattaneo and Garbagnati construction

In [CG16] A. Cattaneo and A. Garbagnati generalized the Borcea-Voisin construction allowing a non-symplectic automorphisms of a $K3$ surfaces of higher orders — 3, 4 and 6.

Theorem 4.2.1 (Cattaneo-Garbagnati, 2013). *Let S be a $K3$ surface admitting a purely non-symplectic automorphism α_S of order n. Let E be an elliptic curve admitting an automorphism α_E such that $\alpha_E(\omega_E) = \zeta_n \omega_E$, where ζ_n denotes n-th root of unity. Then $n = 2, 3, 4, 6$ and $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$, is a singular variety which admits a desingularization which is a Calabi-Yau manifold.*

Proof. The assumption for $n = 2, 3, 4, 6$ is necessarily by [Sil09].

Varieties S and E are Calabi-Yau, thus variety $S \times E$ has trivial canonical bundle and a generator of $H^{3,0}(S \times E, \mathbb{C})$ is $\omega_S \wedge \omega_E$. We have

$$\alpha_S \times \alpha_E^{n-1}(\omega_S \wedge \omega_E) = \zeta_n \cdot \zeta_n^{n-1} \omega_S \wedge \omega_E = \omega_S \wedge \omega_E,$$

hence by 3.1.4 there exists a crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$.

Since the Hodge numbers of Calabi-Yau manifolds are birational invariants (2.1.6), we obtain that a desingularization of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ has the Hodge diamond of a Calabi-Yau type. □

Definition 4.2.2. Any crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$, will be called a *Calabi-Yau 3-fold of Borcea-Voisin type.*

The authors gave a detailed crepant resolution and computed the Hodge numbers of the resulting algebraic varieties. Their approach is based on a general idea: Let

$$X_n := \widetilde{(S \times E)/(\alpha_S \times \alpha_E^{n-1})},$$

be a Calabi-Yau threefold of Borcea-Voisin type. The numbers $h^{1,1}(X_n)$ and $h^{2,1}(X_n)$ depend on the action of $\alpha_S \times \alpha_E^{n-1}$ on $S \times E$. They are the sum of two contributions: one comes from the desingularization of the singular locus of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$, the other comes from the cohomology of $S \times E$ which is invariant for $\alpha_S \times \alpha_E^{n-1}$. Since the fixed loci of α_E^i ($i \in \{1, 2, \ldots, n\}$) are uniquely determined by n, the fixed loci of $\alpha_S \times \alpha_E^{n-1}$ depends only on the properties of the fixed loci of α_S^i.

The construction of a crepant resolution of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$ is based on commutativity of the following diagram

$$\begin{array}{ccc} S \times E & \xleftarrow{\quad\sigma\quad} & \widetilde{S \times E} \\ \Big\downarrow{\scriptstyle \pi} & \cdot & \Big\downarrow{\scriptstyle \pi'} \\ (S \times E)/(\alpha_S \times \alpha_E^{n-1}) & \longleftarrow & \widetilde{(S \times E)/(\alpha_S \times \alpha_E^{n-1})} \end{array}$$

where σ is a blow up in the fixed locus $\mathrm{Fix}(\alpha_S \times \alpha_E^{n-1})$ and π, π' are quotient maps. After blowing up the fixed locus and then considering the quotient by the induced automorphism we obtain a desingularization of $(S \times E)/(\alpha_S \times \alpha_E^{n-1})$.

The part of the cohomology which comes from the cohomology of $S \times E$ may be computed from the Künneth formula 1.0.5 while the affect of singular locus is given by the formula 1.0.7.

4.2.1 Computations of Hodge numbers of X_n

We shall reproof formulas for the Hodge numbers of Calabi-Yau threefolds of Borcea-Voisin type, using the orbifold cohomology formula and the orbifold Euler characteristic introduced in the third chapter.

Order 2

Let (S, α_S) be a $K3$ surface admitting a non-symplectic involution α_S. Consider an elliptic curve E with non-symplectic involution α_E (any elliptic curve E admits such an automorphism). Let us denote by $H^2(S,\mathbb{C})^{\alpha_S}$ the invariant part of cohomology $H^2(S,\mathbb{C})$ under α_S and by r the dimension $r = \dim H^2(S,\mathbb{C})^{\alpha_S}$. We also denote the eigenspace for -1 of the induced action α_S^* on $H^2(S,\mathbb{C})$ by $H^2(S,\mathbb{C})_{-1}$ and by m the dimension $m = \dim H^2(S,\mathbb{C})_{-1}$.

We have decomposition:

$$H^2(S,\mathbb{C}) = H^2(S,\mathbb{C})^{\alpha_S} \oplus H^2(S,\mathbb{C})_{-1},$$

hence the Hodge diamonds of the respective eigenspaces have the following shape:

$$H^{i,j}(S,\mathbb{C})^{\alpha_S} \qquad\qquad H^{i,j}(S,\mathbb{C})_{-1}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & r & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad\qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & m-2 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

The Hodge diamonds of eigenspaces of the induced action of α_E^* on $H^1(E,\mathbb{C})$ have forms

$$H^{i,j}(E,\mathbb{C})^{\alpha_E} \qquad\qquad H^{i,j}(E,\mathbb{C})_{-1}$$

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} \qquad\qquad \begin{array}{ccc} & 0 & \\ 1 & & 1 \\ & 0 & \end{array}$$

By Künneth's formula the Hodge diamond of $H^3(S\times E,\mathbb{C})^{C_2}$ is given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & r+1 & & 0 & \\ 1 & & m-1 & & m-1 & & 1 \\ & 0 & & r+1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

The local action of -1 on a fixed curve may be linearized to matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with age equal to 1. Thus by 3.3.1

$$h^{i,j}(\widetilde{S \times E/C_2}) = h^{i,j}_{\mathrm{orb}}(S \times E/C_2) = h^{i,j}(S \times E)^{C_2} \oplus \bigoplus_{U \in \Lambda(-1)} h^{i-1,j-1}(U),$$

which gives formulas

$$h^{1,1}(\widetilde{S \times E/C_2}) = r + 1 + 4N \quad \text{and} \quad h^{2,1}(\widetilde{S \times E/C_2}) = m - 1 + 4N'. \tag{2-1}$$

Since the quotient S/α_S is a smooth surface we can use stringy number to compute its Euler characteristic:

$$e(S/\alpha_S) = \frac{1}{2}\left(e(S) + 2N - 2N'\right) = 12 + N - N'.$$

On the other hand from the Hodge diamond of S/α_S follows that $e(S/\alpha_S) = r + 2$, thus $r = 10 + N - N'$. Moreover $r + m - 2 = 20$, hence $m = 12 - N + N'$. Putting formulas for r and m into 2-1 we recover formulas from Theorem 1.1 of [CG16].

Order 3

Let (S, α_S) be a $K3$ surface admitting a purely non-symplectic automorphism α_S of order 3. Eigenvalues of induced mapping α_S^* on $H^2(S, \mathbb{C})$ belong to $\{1, \zeta_3, \zeta_3^2\}$. Let us denote by $H^2(S, \mathbb{C})_{\zeta_3^i}$ the eigenspace of the eigenvalue ζ_3^i. For $i = 1, 2$ the dimension of $H^2(S, \mathbb{C})_{\zeta_3^i}$ does not depend on i and will be denoted by m. Moreover let r be the dimension of $H^2(S, \mathbb{C})^{\alpha_S}$ — invariant part of $H^2(S, \mathbb{C})$ under α_S.

Consider an elliptic curve E with the Weierstrass equation $y^2 = x^3 + 1$ together with a non-symplectic automorphism α_E of order 3 such that $\alpha_E(x, y) = (\zeta_3 x, y)$. We denote an automorphism $\alpha_S \times \alpha_E^2$ by α.

We see that

$$H^2(S, \mathbb{C}) = H^2(S, \mathbb{C})^{\alpha_S} \oplus H^2(S, \mathbb{C})_{\zeta_3} \oplus H^2(S, \mathbb{C})_{\zeta_3^2}.$$

Because $\alpha_S^*\big|_{H^{2,0}(S,\mathbb{C})}([\omega_S]) = \zeta_3[\omega_S]$, we get $H^{2,0}(S) \subset H(S, \mathbb{C})_{\zeta_3}$.
The complex conjugation yields $H^{0,2}(S) \subset H^2(S, \mathbb{C})_{\zeta_3^2}$. Finally

$$\alpha^*([\omega_S \wedge \omega_E]) = \overline{\zeta_3}\zeta_3[\omega_S \wedge \omega_E] = [\omega_S \wedge \omega_E],$$

hence the Hodge diamonds of the respective eigenspaces have the following forms

$$\begin{array}{ccc} H^{i,j}(S,\mathbb{C})^{\alpha_S} & H^{i,j}(S,\mathbb{C})_{\zeta_3} & H^{i,j}(S,\mathbb{C})_{\zeta_3^2} \\ \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & r & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} & \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & m-1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} & \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & m-1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \end{array}$$

Similar analysis gives the Hodge diamonds of eigenspaces of action on the Hodge groups.

$$\begin{array}{ccc} H^{i,j}(E,\mathbb{C})^{\alpha_E^2} & H^{i,j}(E)_{\zeta_3} & H^{i,j}(E)_{\zeta_3^2} \\ \begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array} & \begin{array}{ccc} & 0 & \\ 1 & & 0 \\ & 0 & \end{array} & \begin{array}{ccc} & 0 & \\ 0 & & 1 \\ & 0 & \end{array} \end{array}$$

By Künneth's formula the Hodge diamond of the invariant part of $H^3(S \times E, \mathbb{C})$ has the same form as in the case of order 2.

We denote the automorphism $\alpha_S \times \alpha_E^2$ by α. Let us now consider possible actions of elements of $\langle \alpha \rangle \simeq C_3$ on S and E.

The action of α. The action of the automorphism α on E is given by

$$E \ni (x, y) \mapsto (\zeta_3^2 x, y) \in E,$$

hence it has three fixed points. Locally the action of α on components of the fixed locus can be diagonalized to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \zeta_3^2 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}.$$

It follows that ages are equal to 1 and 2 respectively.

The action of α^2. Analogously we gets possible diagonalised matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix},$$

with ages 1 and 1.

Thus decomposing $(S \times E)^\alpha = \mathcal{C} \cup \mathcal{R} \cup \mathcal{P}$, where

$$\mathcal{C} := \{3 \text{ curves with highest genus } g(C)\},$$

$$\mathcal{R} := \{3k-3 \text{ rational curves}\},$$

$$\mathcal{P} := \{3n \text{ isolated points}\},$$

the orbifold formula implies that

$$\begin{aligned} H^{i,j}_{\text{orb}}(S \times E/C_3) &= H^{i,j}(S \times E)^{C_3} \oplus \bigoplus_{U \in \Lambda(\zeta_3)} \bigoplus_{\iota=1}^{2} H^{i-\text{age}(\alpha^\iota),\, j-\text{age}(\alpha^\iota)}(U) = \\ &= H^{i,j}(S \times E)^{C_3} \oplus \left(\bigoplus_{U \in \mathcal{C}} H^{i-1,\, j-1}(U) \oplus H^{i-1,\, j-1}(U) \right) \oplus \\ &\oplus \left(\bigoplus_{U \in \mathcal{R}} H^{i-1,\, j-1}(U) \oplus H^{i-1,\, j-1}(U) \right) \oplus \left(\bigoplus_{U \in \mathcal{P}} H^{i-1,\, j-1}(U) \right). \end{aligned}$$

Therefore by 3.2.3

$$h^{1,1}(\widetilde{X/C_3}) = r + 1 + 6 \cdot 1 + 2 \cdot (3k-3) \cdot 1 + 3n \cdot 1 = r + 1 + 3n + 6k.$$

$$h^{1,2}(\widetilde{X/C_3}) = m - 1 + 2 \cdot 3 \cdot g(C) + (3k-3) \cdot 2 \cdot 0 + 3n \cdot 0 = m - 1 + 6g(C).$$

Hence we proved the following theorem:

Theorem 4.2.3. *If S^{α_S} consists of k curves together with a curve with highest genus $g(C)$ and n isolated points, then for any crepant resolution of the variety $(S\times E)/(\alpha_S \times \alpha_E^2)$ the following holds*

$$h^{1,1} = r + 1 + 3n + 6k \quad \text{and} \quad h^{2,1} = m - 1 + 6g(C).$$

Order 4

Let (S, α_S) be a $K3$ surface with purely non-symplectic automorphism α_S of order 4. Consider an elliptic curve E with the Weierstrass equation $y^2 = x^3 + x$ together with a non-symplectic automorphism α_E of order 4 such that

$$\alpha_E(x, y) = (-x, iy).$$

Additionally, suppose that $S^{\alpha_S^2}$ is not a union of two elliptic curves.

We shall keep the notation of [CG16].

$X = S \times E$,

P – the infinity point of E,

$r = \dim H^2(S, \mathbb{C})^{\alpha_S}$,

$m = \dim H^2(S, \mathbb{C})_{\zeta_6^i}$ for $i \in \{1, 2, \ldots, 5\}$,

N – number of curves which are fixed by α_S^2,

k – number of curves which are fixed by α_S (curves of the first type).

b – number of curves which are fixed by α_S^2 and are invariant by α_S (curves of the second type),

a – number of pairs (A, A') of curves which are fixed by α_S^2 and $\alpha_S(A) = A'$ (curves of the third type),

D – the curve of the highest genus in $S^{\alpha_S^2}$,

n_1 – number of points which are fixed by α_S not laying on the curve D,

n_2 – number of points which are fixed by α_S laying on the curve D.

For the same reasons as in the previous cases

$$h^{1,1}_{\text{orb}}(X)^{C_4} = r + 1 \quad \text{and} \quad h^{2,1}_{\text{orb}}(X)^{C_4} = m - 1.$$

We denote an automorphism $\alpha_S \times \alpha_E^3$ by α. Contrary to the case of an action of a cyclic group of prime order here we have to consider the (non-trivial) action of a stabilizer of an element g (generating a proper subgroup) on the cohomology groups of $\Lambda(g)$. To this end for any $g \in C_4$ let

$$M_g := \bigoplus_{U \in \Lambda(g)} H^{1-\text{age}(g),\, 1-\text{age}(g)}(U).$$

If $g \neq 1$, then $H^{1-\text{age}(g),1-\text{age}(g)}(U) = \mathbb{C}$ when $\text{age}(g) = 1$ and $H^{1-\text{age}(g),1-\text{age}(g)}(U) = 0$ otherwise, so we can identify M_g with the vector space spanned by irreducible components $U \in \Lambda(g)$ with $\text{age}(g) = 1$. We will consider all possible cases.

The action of α and α^3. The action of automorphism α on E is given by

$$E \ni (x, y) \mapsto (-x, -iy) \in E,$$

hence it has two fixed points — P and $(0, 0)$. The fixed locus of α on S consists of k curves and $n_1 + n_2$ isolated points. Since locally the action of α on X along a fixed curve can be diagonalised to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix},$$

we infer that its age equals 1. Near a fixed point we have a matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix},$$

hence the age equals 2.

In case of the action of α^3 on S we observe that the fixed locus consists of k curves and $n_1 + n_2$ points with ages 1. We see that the contribution of $h^{1,1}_{\text{orb}}$ from both actions is equal to $2k + 2k + 2(n_1 + n_2) = 4k + 2(n_1 + n_2)$.

Element of C_4	α	α^3
Irreducible components	$2k$ curves, $2n_1 + 2n_2$ points	$2k$ curves, $2n_1 + 2n_2$
The age	curve: 1, point: 2	curve: 1, point: 1
Contribution to $h^{1,1}_{\text{orb}}$	$2k$	$2k + 2(n_1 + n_2)$

The action of α^2. The automorphism α^2 acts on E as

$$E \ni (x, y) \mapsto (x, -y) \in E,$$

hence it has four fixed points — P, $(0, 0)$, $(i, 0)$, $(-i, 0)$ from which only two are invariant under the action of α and the other two are permuted. After identifying $M_{\zeta_4^2}$ with the vector space spanned by irreducible components of the fixed locus of α^2 we will find the action of induced map α^* on it.

Because the matrix of the action of ${\alpha_E^2}^*$ on the vector space spanned by the fixed points of α_E is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

it follows that it has 3-dimensional eigenspace for $+1$ and 1-dimensional eigenspace for -1.

The fixed locus of α_S^2 consists of N curves with a pairs permuted by α_S. Hence α_S^* on the vector space spanned by the curves fixed by α_S^2 has $(N-a)$-dimensional eigenspace for $+1$ and a-dimensional eigenspace for -1. One can see that $N = k + b + 2a$, so the total contribution to $h^{1,1}_{\text{orb}}$ equals $3(N - a) + a = 3k + 3b + 4a$.

Element of C_4	α^2
Irreducible components	$4N$ curves
The age	curves: 1
Contribution to $h^{1,1}_{\text{orb}}$	$3k + 3b + 4a$

From the orbifold formula follows that

$$h^{1,1}_{\text{orb}} = r + 1 + 4k + 2(n_1 + n_2) + 3k + 3b + 4a = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a.$$

In order to compute $h^{1,2}_{\text{orb}}$ we will use the orbifold Euler characteristic. In the table below we collect all possible intersections $X^g \cap X^h$, where $(g, h) \in C_4^2$.

	1	ζ_4	ζ_4^2	ζ_4^3
1	X	X^{ζ_4}	$X^{\zeta_4^2}$	X^{ζ_4}
ζ_4	X^{ζ_4}	X^{ζ_4}	X^{ζ_4}	X^{ζ_4}
ζ_4^2	$X^{\zeta_4^2}$	X^{ζ_4}	$X^{\zeta_4^2}$	X^{ζ_4}
ζ_4^3	X^{ζ_4}	X^{ζ_4}	X^{ζ_4}	X^{ζ_4}

By 3.3.1 we obtain the formula

$$e(\widetilde{X/C_4}) = e_{\text{orb}}(X/C_4) = \frac{1}{4}\left(12e(X^{\zeta_4}) + 3e(X^{\zeta_4^2})\right) = 6e(S^{\zeta_4}) + 3e(S^{\zeta_4^2}).$$

If D is of the first type, then by the Riemann-Hurwitz formula we infer that

$$e(S^{\zeta_4}) = 2 - 2g(D) + 2(k-1) + n_1 \quad \text{and} \quad e(S^{\zeta_4^2}) = 2 - 2g(D) + 2(N-1).$$

Since $\widetilde{X/C_4}$ is Calabi-Yau we get

$$h^{1,2}(\widetilde{X/C_4}) = h^{1,1}(\widetilde{X/C_4}) - \frac{1}{2}e(\widetilde{X/C_4}) = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a-$$
$$-(-9g(D) + 6k + 3N + 3n_1) = 1 + r + k + 3b + 2n_2 - n_1 + 4a + 9g(D) - 3N.$$

By ([AS15], Thm. 1.1) and ([AS15], Prop. 1) we have the following relations

$$r = \frac{1}{2}(12 + k + 2a + b - g(D) + 4h) \quad \text{and} \quad m = \frac{1}{2}(12 - k - 2a - b + g(D)),$$

where $h = \sum_{C \subseteq S^{\alpha_S}} (1 - g(C))$. Moreover since D is of the first type $h = k - g(D)$, $n_2 = 0$, $n_1 = 2h + 4$ and $b = \frac{n_1}{2}$, thus

$$h^{1,2}(\widetilde{X/C_4}) = 1 + r + k + 3b + 2n_2 - n_1 + 4a + 9g(D) - 3N = m - 1 + 7g(D).$$

If D is of the second type, then analogously the Riemann-Hurwitz formula yields

$$e(S^{\zeta_4}) = 2h + n_1 + n_2 \quad \text{and} \quad e(S^{\zeta_4^2}) = 2 - 2g(D) + 2(N-1).$$

Thus

$$h^{1,2}(\widetilde{X/C_4}) = h^{1,1}(\widetilde{X/C_4}) - (6h + 3n_1 + 3n_2 + 3N - 3g(D)) =$$
$$= 1 + r + 7k + 3b - n_1 - n_2 + 4a - 6h - 3N + 3g(D).$$

Using the additional relations $h = k$, $n_1 + n_2 = 2h + 4$ and $b = \frac{n_1}{2} + 1$, we get

$$h^{1,2}(\widetilde{X/C_4}) = m + 2g(D) - \frac{n_2}{2}.$$

Hence we proved the following theorem:

Theorem 4.2.4 ([CG16], Proposition 6.3)**.** *If $S^{\alpha_S^2}$ is not a union of two elliptic curves, then for any crepant resolution of variety X/C_4 the following formulas hold*

- *If D is of the first type, then*

$$h^{1,1} = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a,$$
$$h^{1,2} = m - 1 + 7g(D).$$

- *If D is of the second type, then*

$$h^{1,1} = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a,$$
$$h^{1,2} = m + 2g(D) - \frac{n_2}{2}.$$

Order 6

Let (S, γ_S) be a $K3$ surface with purely non-symplectic automorphism γ_S of order 6. Consider an elliptic curve E with the Weierstrass equation $y^2 = x^3 + 1$ together with a non-symplectic automorphism γ_E of order 6 such that

$$\gamma_E(x, y) = (\zeta_6^2 x, -y).$$

We shall keep the notation of [CG16].

$X = S \times E$,

P – the infinity point of E,

$r = \dim H^2(S, \mathbb{C})^{\gamma_S}$,

$m = \dim H^2(S, \mathbb{C})_{\zeta_6^i}$ for $i \in \{1, 2, \ldots, 5\}$,

l – number of curves fixed by γ_S,

k – number of curves fixed by γ_S^2,

N – number of curves fixed by γ_S^3,

$p_{(2,5)} + p_{(3,4)}$ – number of isolated points fixed by γ_S of type $(2, 5)$ and $(3, 4)$ i.e. the action of γ_S near the point linearises to respectively

$$\begin{pmatrix} \zeta_6^2 & 0 \\ 0 & \zeta_6^5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta_6^3 & 0 \\ 0 & \zeta_6^4 \end{pmatrix},$$

n – number of isolated points fixed by γ_S^2,

$2n'$ – number of isolated points fixed by γ_S^2 and switched by γ_S,

a – number of triples (A, A', A'') of curves fixed by γ_S^3 such that $\gamma_S(A) = A'$ and $\gamma_S(A') = A''$,

b – number of pairs (B, B') of curves fixed by γ_S^2 such that $\gamma_S(B) = B'$,

D – the curve with the highest genus in the fixed locus of γ_S,

G – the curve with the highest genus in the fixed locus of γ_S^2,

F_1, F_2 – the curves with the highest genus in the fixed locus of γ_S^3,

Remark 4.2.5. From ([Dil12b], Thm. 4.1) follows that $g(D) \in \{0, 1\}$. Moreover by [AST11] we see that if $g(F_1) \neq 0$, $g(F_2) \neq 0$, then $g(F_1) = g(F_2) = 1$. Clearly if $g(D) = 1$, then $D = G \in \{F_1, F_2\}$.

Denote by γ an automorphism $\gamma_S \times \gamma_E^5$. Clearly $\langle\gamma\rangle \simeq C_6$. Similar computations as in the previous cases imply that

$$h^{1,1}_{\mathrm{orb}}(X)^{C_6} = r + 1 \quad \text{and} \quad h^{2,1}_{\mathrm{orb}}(X)^{C_6} = m - 1.$$

For any $g \in C_6$ let

$$M_g := \bigoplus_{U \in \Lambda(g)} H^{1-\mathrm{age}(g),\, 1-\mathrm{age}(g)}(U).$$

The action of γ and γ^5. The action of an automorphism γ on E is given by

$$E \ni (x, y) \mapsto (\zeta_6^4 x, -y) \in E$$

while γ^5 acts on E by

$$E \ni (x, y) \mapsto (\zeta_6^2 x, -y) \in E,$$

thus both have only one fixed point — P.

The fixed locus of γ on S consists of l curves and $p_{(2,5)} + p_{(3,4)}$ isolated points. Locally the action of γ on $S \times E$ along a fixed curve can be diagonalised to a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_6 & 0 \\ 0 & 0 & \zeta_6^5 \end{pmatrix},$$

with age equal to 1. In the fixed point of type $(2,5)$ and $(3,4)$ we get respectively a matrices

$$\begin{pmatrix} \zeta_6^2 & 0 & 0 \\ 0 & \zeta_6^5 & 0 \\ 0 & 0 & \zeta_6^5 \end{pmatrix} \text{ and } \begin{pmatrix} \zeta_6^3 & 0 & 0 \\ 0 & \zeta_6^4 & 0 \\ 0 & 0 & \zeta_6^5 \end{pmatrix},$$

hence their ages equal 2.

In the case of the action of γ^5 on S we observe that the fixed locus consists of l curves and $p_{(2,5)} + p_{(3,4)}$ isolated points. Along the curve we have a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_6^5 & 0 \\ 0 & 0 & \zeta_6 \end{pmatrix},$$

while in fixed points we have a matrices

$$\begin{pmatrix} \zeta_6 & 0 & 0 \\ 0 & \zeta_6^4 & 0 \\ 0 & 0 & \zeta_6 \end{pmatrix} \text{ or } \begin{pmatrix} \zeta_6^2 & 0 & 0 \\ 0 & \zeta_6^3 & 0 \\ 0 & 0 & \zeta_6 \end{pmatrix},$$

hence their ages are equal to 1. The above analysis shows that the contribution to $h^{1,1}_{\text{orb}}$ from both actions equals $l + l + p_{(2,5)} + p_{(3,4)} = 2l + p_{(2,5)} + p_{(3,4)}$.

Element of C_6	γ	γ^5
Irr. comp.	l curves, $p_{(2,5)} + p_{(3,4)}$ pts.	l curves, $p_{(2,5)} + p_{(3,4)}$ pts.
The age	curve: 1, point: 2	curve: 1, point: 1
Contribution to $h^{1,1}_{\text{orb}}$	l	$l + p_{(2,5)} + p_{(3,4)}$

The action of γ^2 and γ^4. Actions of automorphisms γ^2 and γ^4 on E are given by

$$E \ni (x,y) \mapsto (\zeta_6^2 x, y) \in E$$

and

$$E \ni (x,y) \mapsto (\zeta_6^4 x, y) \in E,$$

respectively. Thus they have three fixed points — $\{P, (0,1), (0,-1)\}$ from which only P is invariant under γ and the remaining two are switched.

The matrix of the action ${\gamma_E^5}^*$ on the vector space generated by fixed points of γ_E^4 is

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix} \sim \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix}$$

hence it produces a 2-dimensional eigenspace for $+1$ and 1-dimensional eigenspace for -1. We have similar decomposition in the case of the action ${\gamma_E^5}^*$ on the vector space generated by fixed points of γ_E^2.

Notice that both γ^2 and γ^4 have the same fixed points but different ages, the age at a point of a fixed curve is always 1 while for an isolated fixed point we have $\mathrm{age}(\gamma^2) = 2$ and $\mathrm{age}(\gamma^4) = 1$. So we can identify $M_{\zeta_6^2}$ and $M_{\zeta_6^4}$ with the vector space spanned by irreducible curves and irreducible components in $\Lambda(\gamma^2)$, respectively.

The fixed locus of γ_S^2 and γ_S^4 consists of:

- $p_{(2,5)}$ points fixed by γ_S,
- n' pairs of points switched by γ_S,
- $k - 2b$ curves fixed by γ_S,
- b pairs of curves swapped by γ_S.

Thus the action of γ_S on the vector space spanned by irreducible fixed curves has $(k-b)$-dimensional eigenspace for $+1$ and b-dimensional eigenspace for -1 while the action of γ_S on the vector space spanned by all irreducible components of the fixed locus has $(k - b + p_{(2,5)} + n')$-dimensional eigenspace for $+1$ and $(b + n')$-dimensional eigenspace for -1.

Consequently the contribution of the action of γ^2 to $h^{1,1}_{\mathrm{orb}}$ is

$$2(k-b) + b = 2k - b$$

while the contribution of of γ^4 is

$$2(k - b + p_{(2,5)} + n') + b + n' = 2k - b + 2p_{(2,5)} + 3n'.$$

Finally both actions add to $h^{1,1}_{\text{orb}}$

$$2k - b + 2k - b + 2p_{(2,5)} + 3n' = 4k - 2b + 2p_{(2,5)} + 3n'.$$

Element of C_6	γ^2	γ^4
Irreducible components	$3k$ curves, $3n$ points	$3k$ curves, $3n$ points
The age	curve: 1, point: 2	curve: 1, point: 1
Contribution to $h^{1,1}_{\text{orb}}$	$4k - 2b + 2p_{(2,5)} + 3n'$	

The action of γ^3. An automorphism γ^3 acts on E as

$$E \ni (x, y) \mapsto (x, -y) \in E,$$

hence it has four fixed points — $\{P, (1, 0), (\zeta_3, 0), (\zeta_3^2, 0)\}$ from which only P is invariant under γ and the remaining three form a 3-cycle. The action of γ_E on the fixed locus of γ_E^3 preserves one point and permutes the remaining three points so it produces 2-dimensional eigenspace for $+1$ and 1-dimensional eigenspaces for ζ_3 and ζ_3^2.

The action of γ_S on fixed locus of γ_S^3 preserves $N - 3a$ curves and permutes a triples. Thus the action of γ_S^* has $N - 2a$-dimensional eigenspace for $+1$ and a-dimensional eigenspaces for ζ_3 and ζ_3^2.

The contribution to $h^{1,1}_{\text{orb}}$ equals $2(N - 2a) + a + a = 2N - 2a$.

Element of C_6	γ^3
Irreducible components	$4N$ curves
The age	curve: 1
Contribution to $h^{1,1}_{\text{orb}}$	$2N - 2a$

Consequently, by the orbifold formula we see that

$$\begin{aligned} h^{1,1}_{\text{orb}} &= r + 1 + l + l + p_{(2,5)} + p_{(3,4)} + 4k - 2b + 2p_{(2,5)} + 3n' + 2N - 2a = \\ &= r + 1 + 2l + 2N - 2b + 4k - 2a + 3n' + 3p_{(2,5)} + p_{(3,4)}. \end{aligned}$$

By the orbifold cohomology formula we see that the contribution to $h^{1,2}$ have only curves in $\Lambda(g)$ for any $g \in C_6$.

If $g(D) \geq 1$, by 4.2.5 we can assume that $D = G = F_1$. We see that contributions of γ and γ^5 equal to $2g(D)$. The automorphisms γ^2 and γ^4 have three fixed points on E with one 2-point orbit, hence by Künneth's formula the contribution to $h^{1,2}$ in this case equals $4g(D)$. Since γ^3 has four fixed points with 3-points orbit, we find that it's contribution is equal to $2g(D) + g(F_2) + g(F_2/\gamma_S)$. Thus

$$h^{1,2}(\widetilde{X/C_6}) = m - 1 + 8g(D) + g(F_2) + g(F_2/\gamma_S).$$

Now consider the case $g(D) = 0$. By the same argument as above, we see that contribution of γ^2 and γ^4 equals $2g(G) + 2g(G/\gamma_S)$, while the contribution of γ^3 equals $g(F_1) + g(F_1/\gamma_S) + g(F_2) + g(F_2/\gamma_S)$, hence

$$h^{1,2}(\widetilde{X/C_6}) = m - 1 + 2g(G) + 2g(G/\gamma_S) + g(F_1) + g(F_1/\gamma_S) + g(F_2) + g(F_2/\gamma_S).$$

We proved the following theorem:

Theorem 4.2.6 ([CG16], Proposition 7.3)**.** *For any crepant resolution of variety X/C_6 the following formulas hold*

$$\begin{aligned} h^{1,1} &= r + 1 + 2l + 2N - 2b + 4k - 2a + 3n' + 3p_{(2,5)} + p_{(3,4)}, \\ h^{1,2} &= \begin{cases} m - 1 + 8g(D) + g(F_2) + g(F_2/\gamma_S) & \text{if } g(D) \geq 1, \\ m - 1 + 2g(G) + 2g(G/\gamma_S) + g(F_1) + g(F_1/\gamma_S) & \\ +g(F_2) + g(F_2/\gamma_S) & \text{if } g(D) = 0. \end{cases} \end{aligned}$$

Finally, we will compute again $h^{1,2}$, this time using orbifold Euler characteristic. Since it involves different invariants than in the previous approach, we will get a new relation among them (Corollary 4.2.7).

In the table below we collect all possible intersections $X^g \cap X^h$, where $(g, h) \in C_6^2$.

	1	ζ_6	ζ_6^2	ζ_6^3	ζ_6^4	ζ_6^5
1	X	X^{ζ_6}	$X^{\zeta_6^2}$	$X^{\zeta_6^3}$	$X^{\zeta_6^2}$	X^{ζ_6}
ζ_6	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}
ζ_6^2	$X^{\zeta_6^2}$	X^{ζ_6}	$X^{\zeta_6^2}$	X^{ζ_6}	$X^{\zeta_6^2}$	X^{ζ_6}
ζ_6^3	$X^{\zeta_6^3}$	X^{ζ_6}	X^{ζ_6}	$X^{\zeta_6^3}$	X^{ζ_6}	X^{ζ_6}
ζ_6^4	$X^{\zeta_6^2}$	X^{ζ_6}	$X^{\zeta_6^2}$	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}
ζ_6^5	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}	X^{ζ_6}

From 3.3.1 we see that

$$e(\widetilde{X/C_6}) = \frac{1}{6}\left(24e(X^{\zeta_6}) + 8e(X^{\zeta_6^2}) + 3e(X^{\zeta_6^3})\right) = 4e(S^{\zeta_6}) + 4e(S^{\zeta_6^2}) + 2e(S^{\zeta_6^3}).$$

Thus

$$h^{1,2}(\widetilde{X/C_6}) = h^{1,1}(\widetilde{X/C_6}) - 2e(S^{\zeta_6}) - 2e(S^{\zeta_6^2}) - e(S^{\zeta_6^3}).$$

By the Riemann-Hurwitz formula we obtain

$$e(S^{\zeta_6}) = 2(l-1) + 2 - 2g(D) + p_{(2,5)} + p_{(3,4)},$$

$$e(S^{\zeta_6^2}) = 2(k-1) + 1 - g(G) + 1 - g(G/\gamma_S) + n,$$

$$e(S^{\zeta_6^3}) = 2(N-2) + 1 - g(F_1) + 1 - g(F_1/\gamma_s) + 1 - g(F_2) + 1 - g(F_2/\gamma_s)$$

hence after simplifying

$$h^{1,2}(\widetilde{X/C_6}) = r + 1 - 2l - 2b - 2a + 3n' + p_{(2,5)} - p_{(3,4)} - 2n + 4g(D) + 2g(G) +$$
$$+ 2g(G/\gamma_s) + g(F_1) + g(F_1/\gamma_s) + g(F_2) + g(F_2/\gamma_s).$$

Comparing both formulas we get:

Corollary 4.2.7. *With the notation above, the following relation holds*

$$-m + r + 2 - 2l - 2b - 2a + 3n' + p_{(2,5)} - p_{(3,4)} - 2n + 4g(D) = 0.$$

CHAPTER 5

MODULARITY OF DOUBLE OCTICS

In this chapter we discuss the modularity of Calabi-Yau varieties, especially double octic threefolds. We first explain what, we mean by modularity, introducing famous Weil Conjectures and L-series of projective varieties. We shall describe a resolution of singularities of double covers branched along eight planes. Finally, we shall investigate realization of weight 4 cusp forms in rigid double octics.

5.1 Modularity of Calabi-Yau manifolds

5.1.1 Weil conjectures

In this section we shall discuss Weil conjectures, which provide magnificent properties of projective varieties defined over $\mathbb{Q}$ and their prime reductions.

Definition 5.1.1. A projective variety $X \subset \mathbb{P}^n$ is defined over $\mathbb{Q}$ if its homogeneous ideal is generated by polynomials with rational coefficients.

If projective variety is defined over $\mathbb{Q}$ it also has a model defined over $\mathbb{Z}$. Such varieties admit reduction modulo fields i.e. we consider generators of homogeneous ideal of X in the projective space over another field. The reduction of X over the algebraic closure of field k will be denoted by $\overline{X}_k$. In the case of $k = \mathbb{F}_q$, where $q = p^k$

is a prime power, we write simply $\overline{X}_q$. A prime p is called a prime of a *good reduction* if $\overline{X}_q$ is a smooth projective variety, otherwise it is called a prime of a *bad reduction*.

Definition 5.1.2. Let $q = p^k$ be a prime power and $X/\mathbb{F}_q$ a variety. Then *the zeta function* of $X/\mathbb{F}_q$ is defined by

$$Z_q(t) := \exp\left(\sum_{r=1}^{\infty} N_{q^r}\frac{t^r}{r}\right) \in \mathbb{Q}[[t]],$$

where N_{q^r} is the number of $\mathbb{F}_{q^r}$- rational points in the reduction $\overline{X}_q$.

In general it is a very deep problem to compute the zeta function of a given variety. A. Weil in [Wei49] observed series of remarkable conjectures concerning zeta functions; they are called the *Weil Conjectures* although since 1974 they are theorems.

Theorem 5.1.3 (Weil Conjectures)**.** *Let X be a smooth d-dimensional projective variety defined over the field $\mathbb{F}_q$, where $q = p^k$ is a prime power. Then the function $Z_q(t)$ satisfies the following properties:*

1. *(Rationality) The zeta function is rational i.e.*

$$Z_q(t) = \frac{P_q(t)}{Q_q(t)} \quad \text{for some polynomials } P_q, Q_q \in \mathbb{Q}[t].$$

2. *(Functional equation) The function $Z_q(t)$ satisfies a functional equation:*

$$Z_q\left(\frac{1}{q^d t}\right) = \pm q^{de/2} t^e Z_q(t),$$

where e is the self-intersection of the diagonal in $X \times X$.

3. *Analogue of The Riemann hypothesis holds i.e.*

$$Z_q(t) = \frac{P_{1,q}(t) \cdot P_{3,q}(t) \cdot \ldots \cdot P_{2d-1,q}(t)}{P_{0,q}(t) \cdot P_{2,q}(t) \cdot \ldots \cdot P_{2d,q}(t)},$$

where $P_{0,q}(t) = 1 - t$, $P_{2d,q}(t) = 1 - q^d t$ and

$$P_{i,q}(t) = \prod_{j=1}^{b_i}(1 - \alpha_{i,j} t)$$

for $1 \leq i \leq 2d-1$*, where the* $\alpha_{i,j}$ *are algebraic integers of complex absolute value* $|\alpha_{i,j}| = q^{1/2}$*.*

4. *Assume that* X *arises as the reduction of a variety defined over a number field. Then the following holds:*

$$e = \sum_{i=0}^{2d} (-1)^i b_i = \chi(X_{\mathbb{C}}),$$

where b_i *denote the Betti numbers of* $X_{\mathbb{C}}$*.*

A. Weil proved the conjectures in the case of curves. The rationality was proved by Dwork in [Dwo60], the functional equation by Grothendieck ([Gro95]), and the analogue of the Riemann hypothesis (in fact the most difficult part) was proved by Deligne in [Del74, Del80].

5.1.2 Étale cohomology and L series

To work with arithmetic of projective varieties, we need a new cohomology theory. A. Grothendieck developed the *étale cohomology*, which is a key ingredient postulated by A. Weil in the proof of Weil conjectures (P. Deligne used it to prove Riemann hypothesis).

Let X be a d-dimensional, smooth, projective variety defined over an algebraically closed field. For any prime $l \neq \mathrm{char}(X)$ we define a finite dimensional $\mathbb{Q}_l$-vector spaces denoted by $H^i_{\text{ét}}(X, \mathbb{Q}_l)$ (for $i \in \mathbb{N}$), which have a similar properties to singular cohomology in characteristic zero. More precisely, we want the following conditions:

(i) Functoriality with respect to morphisms of smooth projective varieties: for smooth projective varieties X and Y the following diagram commutes:

$$\begin{array}{ccc} H^i_{\text{ét}}(X, \mathbb{Q}_l) & \xleftarrow{f^*} & H^i_{\text{ét}}(Y, \mathbb{Q}_l) \\ {\scriptstyle \mathrm{id}}\downarrow & \cdot & \downarrow{\scriptstyle \mathrm{id}} \\ H^i_{\text{ét}}(X, \mathbb{Q}_l) & \xleftarrow{f^*} & H^i_{\text{ét}}(Y, \mathbb{Q}_l) \end{array}$$

(ii) $H^i_{\text{ét}}(X, \mathbb{Q}_l) = 0$ for $i > 2d$, where $d = \dim(X)$.

(iii) If X is smooth, then all $\mathbb{Q}_l$-vector spaces $H^i_{\text{ét}}(X, \mathbb{Q}_l)$ are finite dimensional.

(iv) Poincaré duality holds i.e. there is an isomorphism $H^{2d}_{\text{ét}}(X, \mathbb{Q}_l) \simeq \mathbb{Q}_l$ and for any $i \leq d$ there is a perfect pairing

$$H^i_{\text{ét}}(X, \mathbb{Q}_l) \times H^{2d-i}_{\text{ét}}(X, \mathbb{Q}_l) \to H^{2d}_{\text{ét}}(X, \mathbb{Q}_l) \simeq \mathbb{Q}_l.$$

(v) The Lefschetz fixed point formula holds: For a morphism $f\colon X \to X$ of a smooth algebraic variety X, such that the fixed locus $\mathrm{Fix}(f)$ is finite and $1-\mathrm{d}f$ is injective at fixed points (or equivalently, fixed points have multiplicity one) the following formula holds:

$$\#\,\mathrm{Fix}(f) = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}\left(f^* \mid H^i_{\text{ét}}(X, \mathbb{Q}_l)\right),$$

where $f^* \mid H^i_{\text{ét}}(X, \mathbb{Q}_l)$ means an induced homomorphism $f^*\colon H^i_{\text{ét}}(X, \mathbb{Q}_l) \to H^i_{\text{ét}}(X, \mathbb{Q}_l)$.

(vi) The Künneth decomposition formula for product of two varieties X and Y holds:

$$H^i_{\text{ét}}(X \times Y, \mathbb{Q}_l) \simeq \bigoplus_{j+k=i} H^j_{\text{ét}}(X, \mathbb{Q}_l) \otimes H^k_{\text{ét}}(Y, \mathbb{Q}_l).$$

(vii) A comparison theorem with singular cohomology holds: if X is a smooth, projective variety over $\mathbb{C}$, then there is isomorphism of $\mathbb{C}$-vector spaces:

$$H^i_{\text{ét}}(X, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C} \simeq H^i(X, \mathbb{C}).$$

Let X be a d-dimensional Calabi-Yau manifold defined over $\mathbb{Q}$. For a given prime p, denote by Frob_p the *Frobenius morphism* i.e.

$$\mathrm{Frob}_p(x_1, x_2, \ldots, x_n) = (x_1^p, x_2^p, \ldots, x_n^p) \quad \text{for any } (x_1, x_2, \ldots, x_n) \in \overline{X_p}.$$

The Lefschetz fixed point formula yields

$$N_{p^r} = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}\left(\left(\mathrm{Frob}_p^r\right)^* \mid H^i_{\text{ét}}(X, \mathbb{Q}_l)\right).$$

We define polynomials

(1-1) $$P_{i,p^r}(t) := \det\left(1 - \left(\mathrm{Frob}_p^r\right)^* t \mid H^i_{\text{ét}}(X, \mathbb{Q}_l)\right).$$

Recall a well known lemma from linear algebra:

Lemma 5.1.4. *Let f be an endomorphism of finite dimensional vector space V. Then*

$$\sum_{n\geq 1} \operatorname{tr}\left(f^n \mid V\right) \frac{t^n}{n} = -\log\det(1 - ft).$$

From 5.1.4 we easily deduce that

$$Z_{p^r}(t) = \frac{P_{1,p^r}(t)\cdot P_{3,p^r}(t)\cdot \ldots \cdot P_{2d-1,p^r}(t)}{P_{0,p^r}(t)\cdot P_{2,p^r}(t)\cdot \ldots \cdot P_{2d,qp^r}(t)},$$

in particular polynomials from (1.1) coincide with polynomials defined in Riemann hypothesis (iii) in 5.1.3.

Now we define i-th L series of X by the following rule:

$$L\left(H^i_{\text{ét}}(\overline{X}, \mathbb{Q}_l), s\right) := (*)\prod_{p\in\mathcal{P}} \frac{1}{P_{i,p}(p^{-s})},$$

where $\mathcal{P}$ is the set of primes of good reduction and $(*)$ denotes suitable Euler factors for the primes of bad reduction. Usually, the d-th L series $L\left(H^d_{\text{ét}}(\overline{X}, \mathbb{Q}_l), s\right)$ we denote by $L(X, s)$. Deligne's proof of the Riemann hypothesis 5.1.3 follows from the following theorem:

Theorem 5.1.5 (Deligne integrality theorem, [Del74, Del80, Esn06])**.** *Let X be a smooth projective variety defined over the field $\mathbb{F}_q$, where $q = p^k$ is a prime power. Then, for all i and $l \neq p$ the eigenvalues of $\left(\mathrm{Frob}_p^k\right)^*$ on the vector space $H^i_{\text{ét}}(\overline{X}_q, \mathbb{Q}_l)$ are algebraic integers of absolute value $q^{i/2}$.*

In our purpose we are mainly interested in third L series of Calabi-Yau threefold X. In that case the Dirichlet expansion of $L(X, s)$ gives coefficients $a_k(X)$ such that

$$L(X, s) = \sum_{k=1}^{\infty} \frac{a_k(X)}{k^s}, \quad a_1(X) = 1, \quad a_p(X) = \operatorname{tr}\left(\mathrm{Frob}_p^* \mid H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_l)\right)$$

and $a_k(X)$ depends on $a_p(X)$ for every prime divisor p of k.

By [Esn06] there exists $\kappa_p(X) \in \mathbb{Z}$ such that

$$\operatorname{tr}\left(\mathrm{Frob}_p^* \mid H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_l)\right) = \kappa_p(X)p \tag{1-2}$$

and by Poincaré duality

$$\operatorname{tr}\left(\mathrm{Frob}_p^* \mid H^4_{\text{ét}}(\overline{X}, \mathbb{Q}_l)\right) = \kappa_p(X)p^2. \tag{1-3}$$

By the Lefschetz fixed point formula we see that

$$N_p = 1 + \operatorname{tr}\left(\mathrm{Frob}_p^* \mid H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_l)\right) - a_p(X) + \operatorname{tr}\left(\mathrm{Frob}_p^* \mid H^4_{\text{ét}}(\overline{X}, \mathbb{Q}_l)\right) + p^3,$$

hence by (1-2) and (1-3) we infer that

$$a_p(X) = 1 + p^3 + (p + p^2)\kappa_p(X) - N_p$$

depends on numbers of $\mathbb{F}_p$-rational points of X.

5.1.3 Modular forms

The group $\mathrm{Sl}_2(\mathbb{Z})$ is called a *full modular group*, this group acts on the Siegel upper half plane $\mathbb{H}$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{H} \ni \tau \to \frac{a\tau + b}{c\tau + d} \in \mathbb{H}.$$

Subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sl}_2(\mathbb{Z}) \colon c \equiv 0 \pmod{N} \right\} \quad \text{for } N \in \mathbb{N}$$

are called *Hecke congruence subgroups* of $\mathrm{Sl}_2(\mathbb{Z})$.

Definition 5.1.6. An unrestricted modular form of *weight* k for $\Gamma_0(N)$ and *level* N is a holomorphic function $f \colon \mathbb{H} \to \mathbb{C}$ such that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ the following holds

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for } \tau \in \mathbb{H}.$$

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, the function f is a periodic function i.e. $f(\tau+1) = f(\tau)$, for $\tau \in \mathbb{H}$. Hence it has a Fourier series expansion:

$$f(\tau) = \sum_{n=-\infty}^{+\infty} c_n q^n, \quad \text{where } q = e^{2\pi i \tau}.$$

An unrestricted modular form f is called a *modular form* if $c_n = 0$ for $n < 0$ and a *cusp form* if additionally $c_0 = 0$.

On the vector space $S_k(\Gamma_0(N))$ of weight k cusp forms for $\Gamma_0(N)$ we can define operators T_p for prime numbers $p \nmid N$, which are called *Hecke operators* (for details see [DS05, Kna92]). One can prove that for distinct primes p, q, operators T_p and T_q commute and all T_p have the same eigenspaces. The form which is an eigenvector of all T_p is called *Hecke eigenform.* Thus for any Hecke eigenform g and prime number $p \nmid N$, there are integers b_p such that $T_p(g) = b_p g$. There exist a minimal number $N' \mid N$ and Hecke eigenform $h \in S_k(\Gamma_0(N'))$ with Fourier expansion:

$$h(\tau) = \sum_{n=1}^{+\infty} c_n q^n, \quad \text{where } c_p = b_p \text{ for all } p \nmid N.$$

The eigenform h such that $c_1 = 1$ is called *normalized Hecke newform.* To a normalized Hecke newform $f = \sum_{n=1}^{+\infty} a_n q^n$ we can associate an L-function

$$L(f, s) := \sum_{n=1}^{+\infty} a_n q^{-s},$$

which is in fact a *Mellin transform* of f.

5.1.4 Modularity of rigid Calabi-Yau threefolds

A lot of research on Calabi-Yau threefolds was motivated by the following conjecture:

Modularity conjecture 5.1.7. *Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$. Then X is modular, i. e. there exists a weight 4 cusp form f for $\Gamma_0(N)$ such that*

$$L(X, s) = L(f, s).$$

Moreover, the level N is divisible only by primes of bad reduction.

In [DM03] L. Dieulefait and J. Manoharmayum proved the modularity conjecture for all rigid Calabi-Yau threefold defined over $\mathbb{Q}$ with some assumptions on primes of bad reduction.

Theorem 5.1.8 (L. Dieulefait, J. Manoharmayum, [DM03])**.** *Let X be a rigid Calabi-Yau threefold defined over $\mathbb{Q}$, and assume that one of the following conditions holds:*

(i) *X has good reduction at* 3 *and* 7*,*

(ii) *X has good reduction at* 5,

(iii) *X has good reduction at* 3 *and the trace of* Frob_3 *on* $H^3_{\text{ét}}(\overline{X}, \mathbb{Q}_l)$ *is not divisible by* 3.

Then X is modular.

Later, L. Dieulefait in [Die05] improved the above result and deduced some stronger modularity results. In particular he observed that any rigid Calabi-Yau threefold defined over $\mathbb{Q}$ with good reduction at 5 is modular.

Finally the conjecture 5.1.7 was obtained in [GY11] from the proof of Serre's conjecture on Galois representations over finite fields given by C. Khare and J.-P. Wintenberger; see [KW09a, KW09b]. However the question asked by B. Mazur and D. van Straten — which modular forms can be realized as a modular form of some rigid Calabi-Yau threefold, is still open.

5.2 Double octics

Let $\pi\colon X \to \mathbb{P}^3$ be a double covering of $\mathbb{P}^3$ branched along an octic surface D. If D is smooth then X is a smooth Calabi-Yau threefold (see [CS98]), if D is singular then X is also singular, and the singularities of X are in one-to-one correspondence with the singularities of D. The singularities of X can be resolved by a sequence of the so called *admissible* blow-ups with smooth centers.

Topology of nodal double solids was studied by C. H. Clemens in [Cle83], where he considered surfaces of even degree as a branch locus. S. Cynk and T. Szemberg in [CS98] defined *octic arrangement* and studied new aspects of double coverings branched along it.

Definition 5.2.1. Let $S_1, S_2, \ldots, S_r$ denote surfaces in $\mathbb{P}^3$ of dimensions d_i $(1 \leq i \leq r)$ respectively, where $d_1 + d_2 + \ldots + d_r = 8$. We call a sum $S := S_1 \cup S_2 \cup \ldots \cup S_r$ an *octic arrangement* if the following conditions hold

(i) for any $1 \leq i \neq j \leq r$ the surfaces S_i and S_j intersect transversally along a smooth irreducible curve $C_{i,j}$ or they are disjoint,

(ii) the curves $C_{i,j}$, $C_{k,l}$ either coincide or intersect transversally.

We say that an irreducible curve $C \subset S$ is a q-fold curve if exactly q of surfaces $S_1, S_2, \ldots, S_r$ pass through it. A point $P \in S$ is called a p-fold point if exactly p of the surfaces $S_1, S_2, \ldots, S_r$ pass through it.

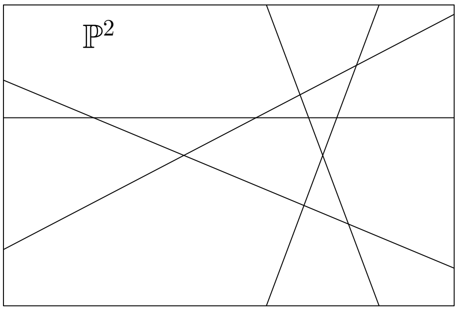
Theorem 5.2.2. *Suppose that octic arrangement S contains only*

(i) *double and triple curves,*

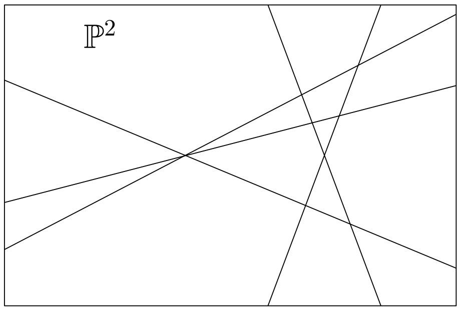
(ii) *q-fold points for $q = 2, 3, 4, 5$,* *then the double cover of* $\mathbb{P}^3$ *branched along* S *has a smooth model, which is a Calabi-Yau threefold.*

Proof. The resolution of singularities is constructed in the following way

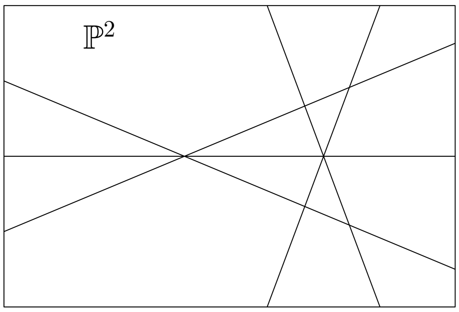
1. *Blow-up of fivefold points.* As the new branch divisor we take the strict transform of the branch divisor plus the exceptional divisor. This introduces five double lines (lying on the exceptional divisor) and a fourfold point on each triple curve passing through that point. The following pictures describe a configuration of lines in the exceptional divisor $\mathbb{P}^2$, given 0, 1 and 2 triple curves at the beginning.



No triple curve

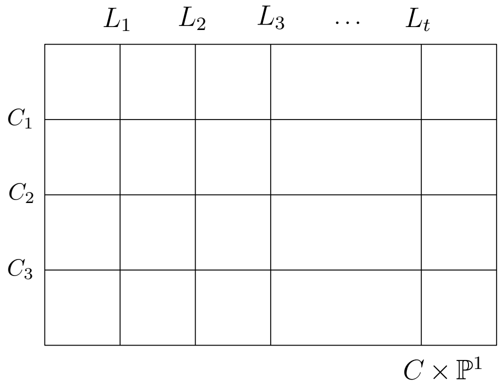


One triple curve

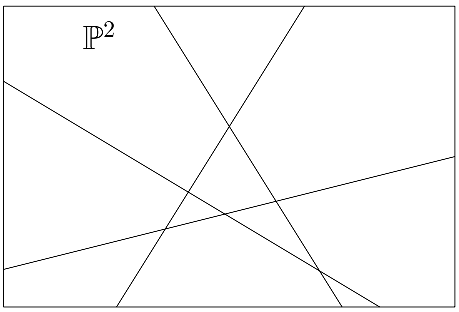


Two triple curves

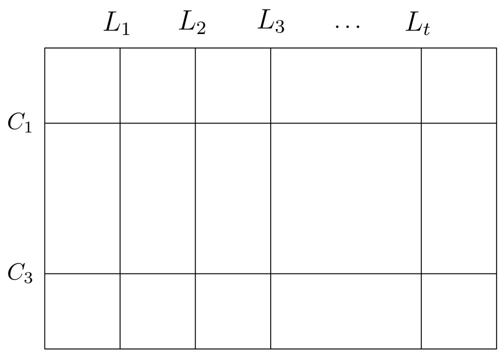
2. *Blow-up of triple curves.* As the new branch divisor we take the strict transform of the branch divisor plus the exceptional divisor. We get three copies C_1, C_2, C_3 of the blown-up curve C as double curves. Moreover every fourfold point lying on that curve gives rise to a double line. Thus in the exceptional divisor $C \times \mathbb{P}^1$, except C_i's, we have also lines $L_1, \ldots, L_t$, where t is the number of fourfold points.



3. *Blow-up of fourfold points.* We take the strict transform of the branch divisor as the new branch divisor. This operation does not introduce new singularities. In the exceptional divisor we get the following configuration:



4. *Blow-up of double curves.* As the new branch divisor we take the strict transform of the branch divisor. Observe that there is no other singularities and arrangements triple points disappear.



Ruled surface over C

□

In our further research we are interested in octic arrangements consisting of eight planes in $\mathbb{P}^3$.

5.3 Rigid realizations of modular forms

5.3.1 Involutions of double octic Calabi-Yau threefolds

Let $S \subset \mathbb{P}^3$ be an octic arrangement consisting of eight planes such that no six intersect and no four contain a line.

A crepant resolution described in 5.2.2 is not uniquely determined but depends on the order of double lines. We can avoid this difficulty replacing the blow-up of all double line separately with the blow-up of their sum in the singular double cover. Now, by the universal property of a blow-up, every automorphism ϕ of the projective space $\mathbb{P}^3$ preserving incidences of eight planes, induces an automorphism $\widetilde{\phi}$ of the double octic X.

Proposition 5.3.1. *Let ϕ be an automorphism of the projective space $\mathbb{P}^3$ of order two such that*

(i) *the fixed locus of ϕ contains no double nor triple line of S,*

(ii) *planes intersecting in fourfold point are not invariant by ϕ,*

(iii) *a fixed line of ϕ intersects S with odd multiplicity in at most two points.*

Then the fixed locus of the induced involution $\widetilde{\phi}\colon X \to X$ contains no curve with positive genus.

Proof. Our strategy is to perform the resolution and after each step verify that the fixed locus of the lifting of ϕ to the partial resolution contains no irrational curves. We start with the (singular) double covering of $\mathbb{P}^3$ branched along S. Any fixed curve of the involution must be mapped to a fixed line of ϕ in $\mathbb{P}^3$, if this curve is irrational then, by the Riemann-Hurwitz formula, the line intersects S with odd multiplicity in at most four points. This contradicts the second assumption, which in fact is necessary.

The next step of the resolution is the blowing up of all fivefold points in the base of double covering, new branch divisor is the strict transform of S plus the exceptional divisors. So any fixed curve of the involution is either birational to a fixed line in previous step or is a fixed line of the induced involution on a projective plane (isomorphic to an exceptional plane of the blow-up). At this step we introduce new double lines: intersections of the exceptional locus with strict transforms of the five planes.

The involution is not the identity on the $\mathbb{P}^2$ because otherwise we will get a fixed divisor in the double octic. If one of the double lines is fixed (dashed line l_1 in the Fig. 1), then the fixed locus on $\mathbb{P}^2$ consists of l_1 and an isolated point. On the other hand on each line $l_2, \ldots, l_4$ we have at least two fixed points, hence the other four lines intersect, which is impossible.

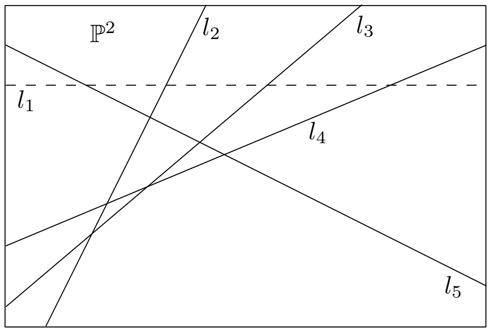
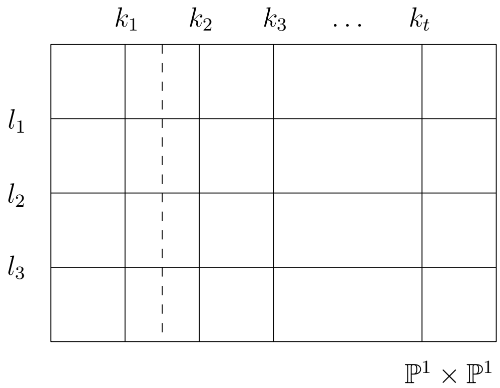


Fig. 1

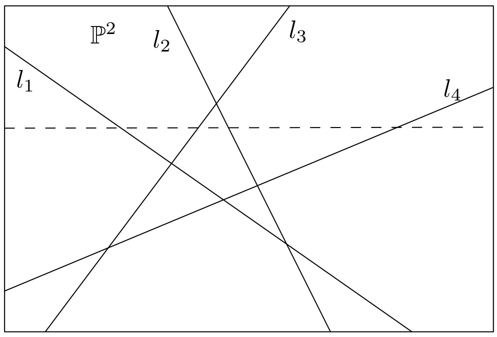
In the next step of resolution we blow up triple lines and as in the case of a fivefold point we add the exceptional divisors to the branch divisor. In the exceptional divisor we get the following configuration:



where t denotes the number of fourfold points on that triple line.

Consequently any fixed curve is a vertical line (dashed line in the picture) and so it is rational. Again we introduce new double lines: l_1, l_2, l_3 and k_1, k_2, $\dots$, k_t (solid lines in the picture). Suppose that the line k_i is fixed. Then we get four invariant planes: three planes containing k_i and one transversal to them, which leads to the contradiction with assumption (ii).

Now, we blow-up all fourfold points, this time we do not add the exceptional divisors to the branch divisor. In the exceptional divisor over a fourfold point P we get the following configuration:



and in the double cover we get a double covering of $\mathbb{P}^2$ branched along the four solid lines. Any irrational fixed curve of the involution maps to a fixed line of involution on $\mathbb{P}^2$ (a dashed line in the picture), it is not of the other lines $l_1, \dots, l_4$ because over each of them the double cover is an isomorphism.

In particular the involution on $\mathbb{P}^2$ is not identity, its fixed locus consists of the dashed line and an isolated point. As an involution of a projective line has two fixed points the lines $l_1, \dots, l_4$ intersect and consequently the four planes passing through the fourfold point intersect along a line, which contradicts our assumptions.

Now the singular locus of the branch divisor is a union of double lines, some of them were introduced in the previous step, but none of them is fixed by the involution. In the last step we blow-up the sum of the double lines in the (singular) cover, a fiber of the blow-up is a line at a double point or a sum of three concurrent lines at a triple points. Since every fixed line is birational to a fixed line from the previous step of the resolution or a line (component of a fiber) the assertion of the proposition follows.

□

5.3.2 Applications

In [Mey05] C. Meyer gave a list of 63 one-parameter families of double octics $\{X_\tau\}_{\tau\in\mathbb{P}^1}$ with $h^{2,1}(X_\tau) = 1$. Using an extensive computer search he found 18 examples in 11 families (see [Mey05], table on page 54) such that for all primes $p \le 97$ the trace of Frob_p equals $a_p + pb_p$, where a_p and b_p are the coefficients of cusp forms f_4 and f_2 of weight 4 and 2, respectively. This gives a strong numerical evidence of the modularity of X in the following sense: the semi-simplification of the Galois representation on $H^3(X)$ decomposes into two-dimensional pieces isomorphic to Galois representations associated to f_4 and f_2. Equivalently, the L series of X factors as $L(X, s) = L(f_4, s)L(f_2, s-1)$ (see [HV05, HV06]). In fact modularity of all examples except Arr. no. 154 was proved in [CM08].

Remark 5.3.2. Different elements from Meyer's table with the same arrangement type are in fact isomorphic with a quadratic twist compatible with twists λ in table [Mey05]. For an Arr. no. 4, from [CC17] we find that the map

$$\begin{pmatrix} x \\ y \\ z \\ t \\ u \end{pmatrix} \mapsto \begin{pmatrix} ABy + ABz \\ -ABy \\ (-A^2 + AB)\, x + (-A^2 + AB)\, y - A^2 z + (A^2 - AB)\, t \\ (AB - B^2)\, t \\ A^3B^3(A-B)^2 u \end{pmatrix}$$

which gives an isomorphism (over $\mathbb{Q}$) between $X_{(A:B)}$ and the quadratic twist of $X_{(A-B:A)}$ by $-A(A-B)$. In particular $X_{(1:-1)}$ and $X_{(1:2)}$ are isomorphic, they are also isomorphic to the quadratic twist of $X_{(2:1)}$ by -2. Since the cusp forms 32k4A1 and 32A1 have CM by $\sqrt{-1}$ the quadratic twists by 2 and -2 coincide. Similar transformations exists also for Arr. no 13, 249 and 267.

Theorem 5.3.3. *Let X be a Calabi-Yau threefold from Meyer's table corresponding to Arr. no. 4, 13, 21, 53, 244, 267, 274. Then there exists a symplectic involution ϕ on X and a crepant resolution Y of a quotient X/ϕ, which is a rigid modular Calabi-Yau threefold.*

Proof. By the remark 5.3.2 for each arrangement type in the theorem we can consider only one example. For all examples, we shall find a transformation $g\colon \mathbb{P}^1 \to \mathbb{P}^1$, which has an isolated fixed point τ_0 such that $g'(\tau_0) = -1$ and isomorphisms $\widetilde{\phi_\tau}\colon X_\tau \to X_{g(\tau)}$ for which ϕ_{τ_0} is a symplectic involution satisfying conditions of 5.3.1.

In the table Tab.1 we give an involution ϕ_τ and the transformation g.

Arr. no.	g	τ_0	$\phi_\tau(x,y,z,t,u)$	Cusp form
4	$1/\tau$	-1	$\big(Ay+Az, -Ay, Ax+Ay, Bt, A^3Bu\big)$	32k4A1
13	$1/\tau$	1	$\big(Bz, By, Bx, At, -AB^3u\big)$	32k4A1
21	$(-1-\tau)/\tau$	$-1/2$	$\big((A+B)x+(A+B)y, (-A-B)y,$ $(-A-B)z, Ax+(A+B)y+Bt,$ $(A+B)^3Bu\big)$	32k4B1
53	$1/\tau$	1	$(Ay, Ax, -Bt, -Bz, A^2B^2u)$	32k4B1
244	$1/\tau$	-1	$(Bx, By, At, Az, -B^2A^2u)$	12k4A1
267	$1/\tau$	-1	$(t, -z, -y, x, u)$	96k4B1
274	$1/\tau$	1	$(z, -t, x, -y, u)$	96k4E1

Tab. 1

Since τ_0 is a fixed point of g the weighted projective transformation ϕ_{τ_0} induces an automorphism $\widetilde{\phi_{\tau_0}}$ of X_{τ_0} that transform ω to $\lambda\omega$ where λ is the quotient of the determinant of the transformation on $\mathbb{P}^3$ by the coefficient in front of u. For each of the cases in the table Tab.1 we compute $\lambda = 1$, so the involution $\widetilde{\phi_{\tau_0}}$ is symplectic.

The action induced by $\widetilde{\phi_{\tau_0}}$ on $H^{1,2}(X_{\tau_0})$ is given, via the Kodaira-Spencer map, by the multiplication by $g'(\tau_0)$, thus $H^{1,2}(X_{\tau_0})^{\langle\widetilde{\phi_{\tau_0}}\rangle} = 0$. The quotient X_{τ_0}/ϕ_{τ_0} admits a crepant resolution Y_{τ_0} by blowing-up of double lines. From the special case of an orbifold formula (see [CR04]) we have the following formula

$$H^{1,2}(Y_{\tau_0}) \simeq H^{1,2}(X_{\tau_0})^{\langle\phi_{\tau_0}\rangle} \oplus \bigoplus_{C\in \mathrm{Fix}(\phi_{\tau_0})} H^{0,1}(C),$$

where $\mathrm{Fix}(\phi_{\tau_0})$ consists of curves, which are fixed by the involution $\widetilde{\phi_{\tau_0}}$. Proposition 5.3.1 yields that Y_{τ_0} is a rigid Calabi-Yau manifold defined over $\mathbb{Q}$. From the involution we get a monomorphism $H^3(Y_{\tau_0}) \to H^3(X_{\tau_0})$, so the modularity (and the cusp form) of Y follows from the modularity of X. □

Remark 5.3.4. According to our knowledge the last two examples in 5.3.3 give first rigid Calabi-Yau realizations of modular forms 96k4B1 and 96k4E1.

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