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ARITHMETIC PROPERTIES OF FINITE QUOTIENTS OF CALABI-YAU TYPE MANIFOLDS

PhD thesis

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Abstract

We construct a series of examples of Calabi-Yau manifolds in arbitrary dimension and compute the main invariants. In particular, we give higher dimensional generalisation of Borcea-Voisin Calabi-Yau threefolds.

We compute Hodge numbers of constructed examples using orbifold Chen-Ruan cohomology. We also give a method to compute a local zeta function using the Frobenius morphism for orbifold cohomology introduced by Rosen.

As an application we observe that some of constructed Calabi-Yau manifolds are unirational with a purely inseparable map from the projective space (Zariski manifold).

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Chapter 1

Introduction

This thesis is devoted to the study of arithmetic properties of Calabi-Yau manifolds obtained as a quotient of Calabi-Yau type variety by an action of a finite group.

Calabi-Yau manifold is a complex, smooth, projective (Kähler) manifold X satisfying

(i) $K_X = \mathcal{O}_X$,

(ii) $H^i\mathcal{O}_X = 0$ for $0 < i < n = \dim X$.

Equivalently

(i') there exists a nowhere vanishing holomorphic n–form on X,

(ii') there are no global holomorphic i–forms on X, for $0 < i < n$.

There are several different definitions of Calabi-Yau varieties, by the famous Calabi-Yau theorem, all definitions are "almost" equivalent. We shall stick to the above definition, in particular we do not assume that a Calabi-Yau manifold is simply connected.

Calabi-Yau manifolds are subject of intensive studies, motivated by their possible applications to physics in the so called *string theory*. String theory predicts that our universe is not just four-dimensional (time-space), but ten-dimensional. The extra six dimensions or complex three "compactify" to a Calabi-Yau variety.

For a compact complex manifold X we define a *Hodge number* of X as the dimension of the Dolbeult cohomology:

$$h^{i,j}(X) = \dim_{\mathbb{C}} H^j\Omega^i(X), \quad \text{for } 0 \le i, j \le n := \dim X.$$

Hodge numbers are very important invariants of a compact complex variety. The Euler characteristic and the Betti numbers may be expressed as

$$i\text{-th Betti number } b_i = \sum_{p+q=i} h^{p,q}(X),$$

$$\text{the Euler characteristic } e(X) = \sum_{0\le p,q\le 2n} (-1)^{p+q} h^{p+q}(X).$$

The Hodge numbers $h^{i,j}(X)$ may be visualized as the *Hodge diamond*, all elliptic curves and all $K3$ surfaces share the same Hodge diamonds

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array} \qquad\qquad \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Elliptic curve $\qquad$ $K3$ surface

The situations is more complicated already in the case of $n \ge 3$, there exists Calabi-Yau threefolds with different Hodge diamonds, in fact we even do not know if the number of Hodge diamonds of Calabi-Yau threefolds is finite.

Serre duality implies that Hodge diamond is invariant under central reflection ($h^{p,q}(X) = h^{n-p,n-q}(X)$). By the Hodge symmetry the Hodge diamond is also invariant under the reflection in the vertical diagonal ($h^{p,q}(X) = h^{q,p}(X)$).

For a Calabi-Yau manifold X of dimension n,

$$h^{0,0}(X) = h^{0,n}(X) = h^{n,0}(X) = h^{n,n}(X) = 1 \quad \text{and} \quad h^{i,0}(X) = h^{0,i}(X) = 0, \quad \text{for} \quad 0 < i < n,$$

hence the Hodge diamond in that case has the following shape:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & & & h^{1,1} & & & & \\ & & \iddots & & & & \ddots & & \\ & 0 & & & \vdots & & & 0 & \\ 1 & & h^{n-1,1} & \cdots & & \cdots & h^{1,n-1} & & 1 \\ & 0 & & & \vdots & & & 0 & \\ & & \ddots & & & & \iddots & & \\ & & & & h^{n-1,n-1} & & & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

A three dimensional Calabi-Yau manifold has two non-obvious (not equal 0 or 1) Hodge numbers

$$h^{1,1}(X) = h^{2,2}(X) = \text{rank Pic}(X) \quad \text{and} \quad h^{2,1}(X) = h^{1,2}(X) = \dim \text{Def}(X),$$

the Hodge diamond of a Calabi-Yau threefold has the following shape

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Based on considerations in the string theory physicists made several predictions concerning properties of Calabi-Yau manifolds, the most famous unsolved mathematical problem inspired by the string theory is the *Mirror Symmetry Conjecture*. There exists several mathematical formulations of the Mirror Symmetry Conjecture, in the simplest version it predicts that for a non-rigid Calabi-Yau threefold X, there exists Calabi-Yau threefold Y such that the Hodge diamonds of X and Y are rotated by 90° i.e.

$$h^{1,1}(X) = h^{1,2}(Y) \quad \text{and} \quad h^{1,2}(X) = h^{1,1}(Y).$$

Calabi-Yau manifolds can be considered as the higher dimensional counterpart of elliptic curves, thus apart from the motivations coming from physics, Calabi-Yau manifolds provide a very interesting framework to study from the point of view of classification of algebraic varieties and arithmetic.

One of the most important method of constricting new examples of Calabi-Yau manifolds are quotients of projective varieties by a finite group. Generally such quotients are often singular and we have to perform a resolution of singularities as a second step of the construction.

If we want to get a Calabi-Yau manifold as a result we have to consider a so called *crepant resolution* of singularities which has no discrepancy in the canonical class. Unfortunately in many natural situations no crepant resolution exists. Moreover if a crepant resolution exists, often it is difficult to construct.

In Chapter 2 we shall collect and introduce necessarily information that we need in further part of the presented thesis. We start by introducing basic properties of the quotient singularities ${}^{\mathbb{C}^n}\!\big/_{G}$, where $G \subset \mathrm{SL}_n(\mathbb{C})$. Then we shall explain in details toric resolution of singularities which provides a method of describing crepant resolution of the quotient singularities in terms of subdivision of the junior simplex. We give explicit examples and discuss known results particularly in dimensions 2, 3 and 4.

Next part of this chapter is devoted to an orbifold cohomology theory for finite quotients. W. Chen and Y. Ruan introduced

$$H^{i,j}_{\mathrm{orb}}\left({}^{X}\!\big/_{G}\right) := \bigoplus_{[g]\in \mathrm{Conj}(G)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{\mathrm{C}(g)},$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G, $\mathrm{C}(g)$ is the centralizer of g, $\Lambda(g)$ denotes the set of irreducible connected components of the fixed points set of $g \in G$ and $\mathrm{age}(g)$ is an integer associated to a matrix corresponding to linearized action of g.

An important application of the Chen-Ruan cohomology is the possibility of computing Hodge numbers of a crepant resolution of singularities of a quotient variety, without referring to an explicit construction of such a resolution. In fact

$$\dim H^{i,j}_{\mathrm{orb}}\left({}^{X}\!\big/_{G}\right) = h^{i,j}\left(\widetilde{{}^{X}\!\big/_{G}}\right),$$

where $\widetilde{{}^{X}\!\big/_{G}}$ is a crepant resolution of finite quotient ${}^{X}\!\big/_{G}$.

Using Chen-Ruan orbifold cohomology we shall find a formula for Hodge numbers of a quotient variety of type ${}^{X_1 \times X_2 \times \ldots \times X_n}\!\big/_{\mathbb{Z}_d^{n-1}}$, where X_i are Calabi-Yau type manifolds with purely non-symplectic $\mathbb{Z}_d$-action.

In the further parts we shall recall *stringy orbifold Euler characteristic* for the quotient of an manifold by the finite group. This formula was introduced by Dixon, Harvey, Vafa and Witten in string theory as the hypothetical correct Euler characteristic formula of a finite quotient. This formula, just like Chen-Ruan cohomology, may be used to compute the Euler characteristic of a crepant resolution of finite quotients.

Then we briefly discuss topological and holomorphic Lefschetz numbers. We shall use them in further parts of thesis to obtain new relation between parameters attached to Calabi-Yau type varieties, especially $K3$ surfaces.

The next section is devoted to study the Zeta function of finite quotients of Calabi-Yau manifolds. The computations of the Zeta function of a variety is much more delicate. Merging Chen-Ruan cohomology with Rosen's result we find an effective methods of computation of the Zeta function of a quotient variety of type $X_1 \times X_2 \times \ldots \times X_n \big/ \mathbb{Z}_d^{n-1}$, where X_i are Calabi-Yau type manifolds with purely non-symplectic $\mathbb{Z}_d$-action.

A large class of arithmetically significant Calabi-Yau threefolds was constructed as a resolution of singularities of a fiber product of rational elliptic surfaces. We end this chapter with a short and necessary introduction to rational elliptic surfaces.

In Chapter three we focus on Cynk-Hulek construction of higher dimensional Calabi-Yau manifolds involving elliptic curve E_d with automorphism of order $d = 2, 3, 4$. Authors proved that the quotient of E_d^n by a group

$$\{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_d^n : m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

which preserves the canonical bundle $\omega_{E_d^n}$ admits a crepant resolution of singularities

$$\widetilde{E_d^n \big/ \mathbb{Z}_d^{n-1}}.$$

Consequently they obtained n-dimensional Calabi-Yau manifold.

The aim of this chapter is to construct a toric crepant resolution of singularities of Calabi-Yau manifold constructed by Cynk-Hulek which uses elliptic curve E_6 admitting purely non-symplectic automorphism of the missing order 6. Iterated approach originally used by Cynk and Hulek cannot be adopted in the case $d = 6$. Moreover using introduced methods in chapter 2, we compute Hodge numbers and local Zeta functions of all resulting Calabi-Yau manifolds.

In the next chapter we shall use Cynk-Hulek construction in order to extend result of Katsura-Schuett to obtain higher dimensional Calabi-Yau manifolds, which are Zariski varieties. As a corollary we obtained in any odd characteristic $p \not\equiv 1 \pmod{12}$ a unirational Calabi-Yau manifold of arbitrary dimension.

The fifth chapter focus on the classical Borcea-Voisin construction of Calabi-Yau threefolds which produces important examples in context of mirror symmetry. Let E be an elliptic curve and let S be a $K3$ surface, both admitting non-symplectic involutions α_E and α_S, respectively, and consider the quotient $S \times E \big/ \alpha_S \times \alpha_E$. Obviously such quotient is singular and have cyclic singularities. A crepant resolution of singularities $\widetilde{S \times E \big/ \alpha_S \times \alpha_E}$ is a Calabi-Yau threefold with Hodge numbers

$$h^{1,1} = 11 + 5N' - N \quad \text{and} \quad h^{1,2} = 11 + 5N - N',$$

where N is the number of curves in $\mathrm{Fix}(\alpha_S)$ and N' denotes the sum of genera of all curves in fixed locus. Voisin used Nikulin's classification to show that constructed threefolds have a "mirror partner" for all but one of the examples with $N' \neq 0$.

A. Cattaneo and A. Garbagnati generalized the Borcea-Voisin construction allowing non-symplectic automorphism of a $K3$ surfaces of higher orders. They computed Hodge numbers of resulting threefolds by careful study of an explicit construction of a crepant resolution.

We shall give higher dimensional generalisation of the classical Borcea-Voisin threefold by considering quotient $S_d \times E_d^{n-1} / G_{d,n}$, for E_d and $K3$ – surface admitting a purely non-symplectic automorphism of order $d = 2, 3, 4, 6$. We compute Hodge numbers and the Zeta function of resulting varieties. Moreover in the last section we discuss other similar constructions for future study.

In the final chapter we discuss Schoen's construction of Calabi-Yau threefolds as a resolution of singularities of the fiber product of rational elliptic surfaces. We briefly discuss known results and give a method of computation of the Hodge numbers of standard Kummer fiberwise construction by using Chen-Ruan cohomology formula.

Summary of the results

The main original results presented in this thesis are the following:

- Formula for Hodge numbers of a crepant resolution (providing it exists) of the quotient $X_1 \times X_2 \times \ldots \times X_n / \mathbb{Z}_d^{n-1}$, where X_i are Calabi-Yau type manifolds with purely non-symplectic $\mathbb{Z}_d$-action (Theorem 2.3.3).

- Formula for the Zeta function of the crepant resolution (providing it exists) of the quotient $X_1 \times X_2 \times \ldots \times X_n / \mathbb{Z}_d^{n-1}$, where X_i are Calabi-Yau type manifolds with purely non-symplectic $\mathbb{Z}_d$-action (Theorem 2.7.1).

- Construction of a toric crepant resolution of singularities of the quotient $E_6^n / \mathbb{Z}_6^{n-1}$, which completes the construction of Cynk-Hulek for elliptic curves E_6 admitting purely non-symplectic automorphism of order 6 (Theorem 3.2.3).

- Two methods of computation of Hodge numbers of Cynk-Hulek varieties $X_{d,n}$ (Section 3.3).

- Computation of the Zeta function of Cynk-Hulek varieties $X_{d,n}$ (Section 3.4).

- Construction of new arbitrary dimensional Zariski Calabi-Yau manifolds in characteristic $p \not\equiv 1 \pmod{12}$ (Theorems 4.3.3, 4.3.5, 4.3.7).
- Higher dimensional generalisations of classical Borcea-Voisin construction, computation of their Hodge numbers and the Zeta function (Chapter 5).
- Introducing new methods of computations of Hodge numbers of Kummer fibrations of rational elliptic surfaces (Chapter 6).

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Chapter 2

Preliminaries

In this chapter we shall collect basic information that we need in further parts of present thesis. We shall recall basic properties of canonical singularities. Then we introduce necessarily background form toric geometry to analyse later existence of a crepant resolution of singularities of the quotient $\mathbb{C}^n/_G$, where G is a finite group acting on $\mathbb{C}^n$. We give explicit examples and collect known results.

Next part of this chapter is devoted to the orbifold cohomology theory for finite quotients introduced by W. Chen and Y. Ruan and stringy Euler characteristic. Also we briefly recall basic properties of the zeta function of an algebraic variety and give explicit examples in dimensions 2 and 3.

We end this chapter by proving formulas for Hodge numbers and Zeta function of a quotient variety of type $X_1 \times X_2 \times \ldots \times X_n/_{\mathbb{Z}_d^{n-1}}$, where X_i are Calabi-Yau type manifolds with purely non-symplectic $\mathbb{Z}_d$-action.

§ 2.1 Toric resolution of singularities

2.1.1 Canonical singularities

Assume X is a normal, quasi-projective variety defined over an algebraically closed field of characteristic 0.

Definition 2.1.1. Pair $(\widetilde{X}, \sigma)$, where $\sigma : \widetilde{X} \to X$ is a proper, regular and birational map and $\widetilde{X}$ is a smooth algebraic variety, is called a *resolution of singularities* of X if

$$\sigma\Big|_{\sigma^{-1}(\mathrm{Reg}(X))} : \sigma^{-1}(\mathrm{Reg}(X)) \to \mathrm{Reg}(X)$$

is an isomorphism, where $\mathrm{Reg}(X)$ denotes the set of regular points of X.

By the famous result of Hironaka ([Hir64]), every algebraic variety over algebraically closed field of characteristic 0 has a resolution of singularities.

Definition 2.1.2. A variety X has *canonical singularities* if and only if it satisfies the following two conditions

(i) the Weil divisor rK_X is Cartier for some integer $r \geq 1$,

(ii) for a resolution $f : \widetilde{X} \to X$ of singularities of X, and all exceptional prime divisors $\{E_i\}_{i=1}^n$ of f we have the following equality

$$rK_{\widetilde{X}} = f^*\left(rK_X\right) + \sum_{i=1}^{n} a_i E_i,$$

where a_i are non-negative rational numbers for $1 \leq i \leq n$.

The smallest $r \geq 1$ for which rK_X is Cartier in a neighbourhood of singular point $P \in X$ is called the *index* of P.

The surface canonical singularities are exactly nonsingular points, together with so called *Du Val surface singularities* (see [Reib] for a proof) i.e. the hypersurface singularities given by the following equations:

$$\begin{aligned}
&\text{Type } A_n: && x^2 + y^2 + z^{n+1} = 0 \quad \text{for } n \geq 1,\\
&\text{Type } D_n: && x^2 + y^2 z + z^{n-1} = 0 \quad \text{for } n \geq 4,\\
&\text{Type } E_6: && x^2 + y^3 + z^4 = 0,\\
&\text{Type } E_7: && x^2 + y^3 + yz^3 = 0,\\
&\text{Type } E_8: && x^2 + y^3 + z^5 = 0.
\end{aligned}$$

Du Val singularities are also known as *simple surface* singularities, *rational double points* or *Kleinian singularities*.

The $\mathbb{Q}$-divisor

$$\Delta_X := \frac{1}{r} \sum_{i=1}^{n} a_i E_i$$

which satisfies $K_Y = f^* K_X + \Delta_X$ is called *discrepancy* of f. In the Minimal Model Program crucial role play two types of singularities:

- X has *terminal singularities* if $a_i > 0$ for $1 \leq i \leq n$,
- X has *log-terminal singularities* if $a_i > -1$ for $1 \leq i \leq n$.

In our constructions we use resolution of singularities $\widetilde{X}$ which is a variety with trivial canonical divisor K_X (Calabi-Yau type condition).

Definition 2.1.3. A resolution $f : \widetilde{X} \to X$ of a canonical singularities is *crepant* if $a_i = 0$ for $1 \le i \le n$ i.e.

$$K_{\widetilde{X}} = f^* K_X.$$

The name "crepant" was coined by M. Reid by removing "dis" from the word "discrepant", to indicate that the resolutions have no discrepancy in the canonical class.

2.1.2 Quotient singularities

Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup. By the Hilbert basis theorem, the ring of invariants

$$\mathbb{C}[X_1, X_2, \dots, X_n]^G$$

is finitely generated. Consider the quotient variety

$$\mathbb{C}^n/_G := \operatorname{Spec} \mathbb{C}[X_1, X_2, \dots, X_n]^G.$$

Definition 2.1.4. Singularities of $\mathbb{C}^n/_G$ are called *quotient singularities*.

Any Du Val surface singularity is isomorphic to $\mathbb{C}^2/_G$, where $G \subset \mathrm{SL}_2(\mathbb{C})$ is a finite group namely G is of one of the following types

Type A_n: $\mathbb{C}^2/_G$, where $G := \mu_{n+1}$ is the cyclic group of order n with generators:

$$\begin{pmatrix} \zeta_{n+1} & 0 \\ 0 & \zeta_{n+1}^n \end{pmatrix}, \quad \text{where } \zeta_{n+1} \text{ is a primitive } (n+1)^{\text{st}} \text{ root of unity.}$$

Type D_n: $\mathbb{C}^2/_G$, where $G := \mathbb{D}_{4(n-2)}$ is the binary dihedral group of order $4(n-2)$ generated by:

$$\mu_{2(n-2)} \quad \text{and} \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \text{where } i = \sqrt{-1}.$$

Type E_6: $\mathbb{C}^2/_G$, where $G := \mathbb{T}_{24}$ is the binary tetrahedral group of order 24 generated by:

$$\mathbb{D}_8 \quad \text{and} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}.$$

Type E_7: $\mathbb{C}^2/_G$, where $G := \mathbb{O}_{48}$ is the binary octahedral group of order 48 generated by:

$$\mathbb{T}_{24} \quad \text{and} \quad \begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}.$$

Type E_8: $\mathbb{C}^2/_G$, where $G := \mathbb{I}_{120}$ is the binary icosahedral group of order 120 generated by:

$$\frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^4-\zeta_5 & \zeta_5^2-\zeta_5^3 \\ \zeta_5^2-\zeta_5^3 & \zeta_5-\zeta_5^4 \end{pmatrix} \quad \text{and} \quad \frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^2-\zeta_5^4 & \zeta_5^4-\zeta_5 \\ 1-\zeta_5 & \zeta_5^3-\zeta_5 \end{pmatrix}.$$

In fact the above list is a complete list of conjugacy classes of all non-trivial finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ (see [LW12]). Moreover, groups $\mathbb{T}_{24}$, $\mathbb{O}_{48}$ and $\mathbb{I}_{120}$ are obtained as the lift of the symmetry group of corresponding Platonic solid under the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$.

Definition 2.1.5. An element $g \in G$ is a *quasi-reflection* if it is of finite order and $g - \mathrm{id}_n$ has rank 1, where id_n denotes the identity matrix $n \times n$.

A group G which does not contain any quasi-reflection is called *small*.

Theorem 2.1.6 ([ST54], [Che55])**.** *The quotient algebraic variety $\mathbb{C}^n/_G$ is smooth iff G is generated by quasi-reflection, and in that case $\mathbb{C}^n/_G \simeq \mathbb{C}^n$.*

Therefore in the study of the quotient singularity $\mathbb{C}^n/_G$, where $G \subseteq \mathrm{GL}_n(\mathbb{C})$ one may restrict to quotient singularities for small groups G.

Definition 2.1.7. An algebraic variety is *Gorenstein* iff it is Cohen-Macaulay and the canonical sheaf ω_X is invertible.

Theorem 2.1.8 ([Rei80])**.** *Gorenstein quotient singularities are canonical.*

Theorem 2.1.9 ([Rei80], [Tai82])**.** *Let $G \subset GL_n(\mathbb{C})$ be a small finite subgroup which acts linearly on $\mathbb{C}^n$. Then $\mathbb{C}^n/_G$ has canonical singularities iff any element $g \in G$ has a fixed order d and the action of g on $\mathbb{C}^n$ has the following linearisation*

$$\begin{pmatrix} \zeta_d^{a_1} & 0 & 0 & \dots & 0 \\ 0 & \zeta_d^{a_2} & 0 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & 0 & \dots & \zeta_d^{a_n} \end{pmatrix}$$

where $0 \le a_i < d$ and $\sum_{i=1}^n a_i \ge d$. Let us denote such an element g by $\frac{1}{d}(a_1, a_2, \dots, a_n)$.

Moreover

- *an element g for which $\sum_{i=1}^n a_i = d$ corresponds to a crepant exceptional divisor of a resolution $\mathbb{C}^n/_G$,*

- ${}^{\mathbb{C}^n}\!/_G$ *has terminal singularities iff* $\sum_{i=1}^{n} a_i > d$ *for every* $g \in G$.

Theorem 2.1.10 ([Hin76], [Wat74]). *Let $G \subset GL_n(\mathbb{C})$ be a small finite subgroup, then the following three conditions are equivalent*

1. ${}^{\mathbb{C}^n}\!/_G$ *is Gorenstein,*

2. $d \mid \sum_{i=1}^{n} a_i$ *for every* $g \in G$,

3. $G \subset \mathrm{SL}_n(\mathbb{C})$.

Definition 2.1.11. Let $g \in \mathrm{GL}_n(\mathbb{C})$ be a matrix of finite order and let $e^{2\pi i a_1}, e^{2\pi i a_2}, \dots, e^{2\pi i a_m}$ be eigenvalues of g for some $a_1, a_2, \dots, a_m \in [0, 1) \cap \mathbb{Q}$. The sum $\sum_{i=1}^{m} a_i$ is called the *age* of g and is denoted by $\mathrm{age}(g)$. Element g with $\mathrm{age}(g) = 1$ is called *junior*.

The age of G is an integer if and only if $\det g = 1$ i.e. $g \in \mathrm{SL}_n(\mathbb{C})$. Moreover

$$\mathrm{age}(g) + \mathrm{age}(g^{-1}) = \mathrm{rank}(g - I_n) \in \mathbb{Z}.$$

The age of the element $g := \frac{1}{d}(a_1, a_2, \dots, a_n)$ (c.f. 2.1.9) is equal to

$$\mathrm{age}(g) = \frac{1}{d} \sum_{i=1}^{n} a_i.$$

2.1.3 Toric resolution of singularities

This section is devoted to introduce necessarily background from toric geometry. For a complete exposition see [Ful93] or [CLS11].

Let N be a free $\mathbb{Z}$-module of rank n and let $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ with canonical basis $e_1, e_2, \dots, e_n$.

Definition 2.1.12. A cone σ in $N_{\mathbb{R}}$ is said to be *strongly convex polyhedral* if it is generated by finitely many elements of the lattice N over $\mathbb{R}_{\geq 0}$.

A cone is called *nonsingular* if it is generated by part of a basis for the lattice.

Let $M := \mathrm{Hom}(N, \mathbb{Z})$ be a dual lattice of N and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. The dual cone

$$\check{\sigma} := \{m \in M_{\mathbb{R}} : \langle m, x \rangle \geq 0 \text{ for all } x \in \sigma\} \subset M_{\mathbb{R}}$$

leads to a finitely generated $\mathbb{C}$-algebra $\mathbb{C}[\check{\sigma} \cap M]$ which determines *affine toric variety*

$$X_{N,\sigma} := \text{Spec } \mathbb{C}[\check{\sigma} \cap M].$$

Moreover $X_{N,\sigma}$ is smooth iff cone σ is nonsingular, in that case

$$X_{N,\sigma} \simeq \mathbb{C}^{\dim(\sigma)} \times \mathbb{C}^{*n-\dim(\sigma)},$$

where $\dim(\sigma)$ denotes the *dimension* of the cone σ i.e. dimension of $\mathbb{R}\sigma$ as vector space over $\mathbb{R}$.

Definition 2.1.13. A *fan* is a finite collection of strongly convex polyhedral cones $C := \{\sigma_i\}_{i\in I} \subset N_{\mathbb{R}}$ such that

1) any *face* of a cone $\sigma \in C$, i.e. $\sigma \cap H_m$ for some $m \in \check{\sigma}$ and $H_m := \{x \in N_{\mathbb{R}} : \langle m, x\rangle = 0\}$, is also a cone in C,
2) any two cones in C intersect in a common face.

Definition 2.1.14. The *toric variety* $X_{N,C}$ associated to a fan C is the union of $X_{N,\sigma}$ for each $\sigma \in C$ with glued pairs $(X_{N,\sigma_1}, X_{N,\sigma_1})$ along $X_{N,\sigma_1\cap\sigma_2}$ for any $\sigma_1, \sigma_2 \in C$.

Properties of the fan determine geometric properties of the corresponding toric variety.

Theorem 2.1.15 ([CLS11], Thm. 11.1.9)**.** *Every fan $C \subset N_{\mathbb{R}}$ has a refinement C' such that*

- *C' is nonsingular (every cone $\sigma \in C'$ is nonsingular),*
- *C' contains every nonsingular cone of C,*
- *The toric morphism $\pi : X_{C',N} \to X_{C,N}$ is a projective resolution of singularities.*

The refinement C' is obtained form C by a sequence of so called star subdivisions.

Now let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite abelian group of order d. Any element of G can be written as $\frac{1}{d}(a_1, a_2, \dots, a_n)$ for some $0 \leq a_i < d$. Let $\phi : \mathbb{C}^n \to {\mathbb{C}^n}/_{G}$ be the quotient map. There exists a morphism of toric varieties

$$\widetilde{\sigma} : X_{\widetilde{N},\sigma} \to X_{N,\sigma}$$

corresponding to ϕ, where $\widetilde{N}$ is the free $\mathbb{Z}$-module of rank n such that $N \subset \widetilde{N}$ and ${\widetilde{N}}/_{N} \simeq G$. Denote by $\widetilde{g} := \frac{1}{d}(a_1, a_2, \dots, a_n)$ a primitive element in $\widetilde{N}$ which is a lift of an element $g = \frac{1}{d}(a_1, a_2, \dots, a_n)$.

For any refinement C' of C by Lemma 11.4.10 in [CLS11] it follows that

$$K_{X_{\widetilde{N},C'}} = \pi^*\left(K_{X_{N,C}}\right) + \sum_{\tau \in C(1)} a_\tau E_\tau,$$

where $C(1)$ denotes the collection of 1 dimensional cones in C, E_τ is an exceptional divisor corresponding to τ, $a_\tau := \mathrm{age}(g_\tau) - 1$, where g_τ is the primitive element in τ.

Therefore if Σ is a subdivision of C spanned by junior points, then if $X_{\Sigma,\widetilde{N}}$ is smooth, then it is a crepant resolution of ${\mathbb{C}^n}/{G}$. In other words: crepant resolution of ${\mathbb{C}^n}/{G}$ can be identified with equal-volume triangulation of

$$\Delta_n := \text{convex hull of } e_1, e_2, \dots, e_n \text{ in } \widetilde{N},$$

by junior points in $\widetilde{N}$, Δ_n is called *junior simplex.*

2.1.4 Crepant resolution of ${\mathbb{C}^n}/{G}$

Dimension $n = 2$

In dimension 2, a crepant resolution of a quotient by a subgroup of $\mathrm{Sl}_2(\mathbb{C})$ always exists and is unique as follows from the classical result of F. Klein ([Kle73]).

Theorem 2.1.16. *Let G be a finite subgroup of* $\mathrm{Sl}_2(\mathbb{C})$*. The surface* ${\mathbb{C}^2}/{G}$ *admits a unique crepant resolution.*

The quotients ${\mathbb{C}^2}/{G}$ are the Kleinian singularities of type A, D, E described in 2.1.2.

Consider surface cyclic quotient singularities. Since we restrict our considerations to small groups, it is enough to restrict to only cyclic quotients of type $\frac{1}{r}(1,a)$ with $\gcd(a,r) = 1$. These singularities may be described by using *Hirzebruch-Jung resolutions* for details see [Reia].

Definition 2.1.17. Let $r > a > 0$ be coprime integers. Then the *Hirzebruch-Jung continued fraction* of $\frac{r}{a}$ is the expression

$$\frac{r}{a} = a_1 - \cfrac{1}{a_2 - \frac{1}{a_3 - \dots - \frac{1}{a_k}}} =: [a_1, a_2, \dots, a_k],$$

for some integers $a_i \geq 2$.

Consider lattice $L := \mathbb{Z}^2 + \frac{1}{r}(1,a)\mathbb{Z}$ containing the lattice $\mathbb{Z}^2$ as a sublattice of index r. It's other cosets are represented by $r-1$ lattice points $\frac{1}{r}(i \pmod{r}, ia \pmod{r})$ contained in the unit square of the positive quadrant spanned by $(0,1)$ and $(1,0)$, we shall denote this cone by σ.

Now take Newton polygon of all non-zero lattice points inside this square:

$$e_0 = (0,1), \quad e_1 = \frac{1}{r}(1,a), \quad e_2, \quad \ldots, \quad e_{k+1} = (1,0). \tag{2.1.1}$$

Then we have the following properties:

- Points e_i, e_{i+1} for $i = 0, \ldots, k$ form an oriented basis of L,
- For any $i = 1, \ldots, k$ we have

  $$e_{i+1} + e_{i-1} = a_i e_i,$$

  where

  $$\frac{r}{a} = [a_1, a_2, \ldots, a_k].$$

To perform the resolution we subdivide the positive quadrant with rays generated by e_0, $e_1, \ldots, e_{k+1}$. Denote by $C \subset C'$ the corresponding fans. The refinement $C \subset C'$ gives a toric morphism $X_{C',\widetilde{L}} \to X_{C,L}$ called *Hirzebruch-Jung resolution*.

Theorem 2.1.18 ([Reia]). *Consider cyclic singularity $\mathbb{C}^2/\mathbb{Z}_r$ of type $\frac{1}{r}(1,a)$. For each $i = 0, 1, \ldots, k$, let $\xi_i(x,y)$, $\eta_i(x,y)$ be monomials forming the basis of*

$$\{(\alpha, \beta) : \ \alpha + a\beta \equiv 0 \pmod r\} \subset \mathbb{Z}^2$$

dual to the basis e_i, e_{i+1} from 2.1.1. *Then $\mathbb{C}^2/\mathbb{Z}_r$ has a resolution of singularities $\widetilde{\mathbb{C}^2/\mathbb{Z}_r}$ such that the map*

$$\mathbb{C}^2 \to \widetilde{\mathbb{C}^2/\mathbb{Z}_r}$$

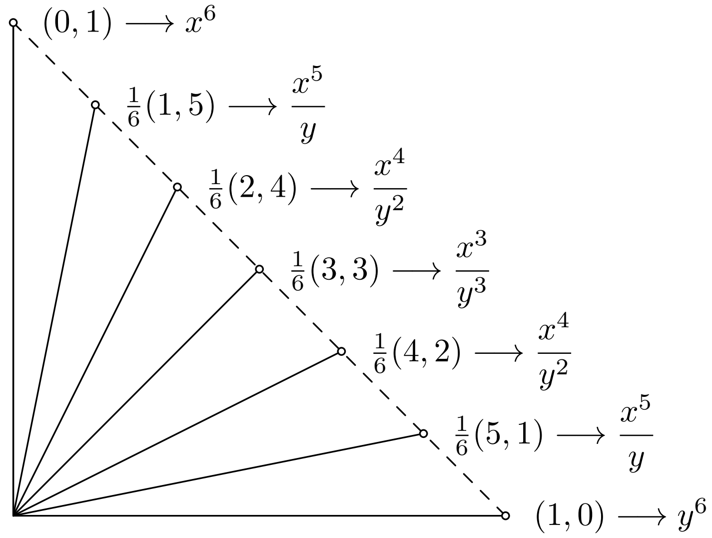
is given in affine charts by

$$(x,y) \to (\xi_i(x,y), \eta_i(x,y)).$$

Remark 2.1.19. If $1 + a = r$, then by 2.1.9 theorem 2.1.18 gives a subdivision of 2-simplex connecting $(0,1)$ and $(1,0)$ into r equal-length segments and hence crepant resolution of singularity $\frac{1}{r}(1,a)$.

Example 2.1.20. Consider the cyclic singularity of type $\frac{1}{6}(1,5)$. We see that $\frac{6}{5} = [2,2,2,2,2]$, the Newton polygon contains junior points

$$e_0 = (0,1), \ e_1 = \frac{1}{6}(1,5), \ e_2 = \frac{1}{6}(2,4), \ e_3 = \frac{1}{6}(3,3), \ e_4 = \frac{1}{6}(4,2), \ e_5 = \frac{1}{6}(5,1), \ e_6 = (1,0).$$



Passing to the lattice of invariant monomials, we mark rational function in each junior point (1-dimensional simplex) which corresponds to a generator of a dual cone $\langle e_i, e_{i+1}\rangle$ for $i = 0, 1, \ldots, 5$. Each rational function (or inverse one) is a coordinate of inverse map of a resolution of $\frac{1}{6}(1,5)$ given in affine charts by

$$\left(x^6, \frac{y}{x^5}\right), \left(\frac{x^5}{y}, \frac{y^2}{x^4}\right), \left(\frac{x^4}{y^2}, \frac{y^3}{x^3}\right), \left(\frac{x^3}{y^3}, \frac{y^4}{x^2}\right), \left(\frac{x^2}{y^4}, \frac{y^5}{x}\right) \text{ or } \left(\frac{x}{y^5}, y^6\right).$$

Dimension 3

In dimension 3 situation is more subtle. A crepant resolution always exist but it is not unique. Any two crepant resolutions are related by a sequence of the so called *flops* (for a details see [Kol89]).

Theorem 2.1.21 ([MOP87], [Roa96])**.** *Let G be a finite subgroup of* $\mathrm{Sl}_3(\mathbb{C})$*. Then* $\mathbb{C}^3/G$ *admits a crepant resolution* $\widetilde{\mathbb{C}^3/G}$*. Moreover*

$$e\left(\widetilde{\mathbb{C}^3/G}\right) = \text{number of conjugacy classes of } G.$$

Roan gave first complete proof of the crepant resolution of singularities in dimension 3 and is based on case by case analysis of the following classification of conjugacy classes of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$:

Type A: A subgroup consisting of matrices

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \quad \text{where } \alpha, \beta, \gamma \in \mathbb{C}.$$

Type B: A subgroup consisting of matrices

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix} \quad \text{where } \alpha, a, b, c, d \in \mathbb{C}.$$

Type C: A subgroup generated by an abelian group $H \in$**Type** A and

$$T := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Type D: A subgroup generated by an abelian groups $H, T \in$ **Type** C and

$$R := \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}, \quad \text{with } a, b, c \in \mathbb{C} \text{ such that } abc = 1.$$

Type E: A subgroup of order 108 generated by T and

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_3 & b \\ 0 & 0 & \zeta_3^2 \end{pmatrix} \quad \text{and} \quad V := \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta_3 & \zeta_3^2 \\ 1 & \zeta_3^2 & \zeta_3 \end{pmatrix}.$$

Type F: A subgroup of order 216 generated by **Type** E and

$$\frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & 1 & \zeta_3^2 \\ 1 & \zeta_3 & \zeta_3 \\ \zeta_3 & 1 & \zeta_3 \end{pmatrix}.$$

Type G: A subgroup of order 648 generated by **Type** E and

$$\begin{pmatrix} \zeta_9^4 & 0 & 0 \\ 0 & \zeta_9^4 & 0 \\ 0 & 0 & \zeta_9 \end{pmatrix}.$$

Type H: Icosahedral group of order 60 generated by T and

$$E_2 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad E_3 := \frac{1}{2} \begin{pmatrix} -1 & \ell_- & \ell_+ \\ \ell_- & \ell_+ & -1 \\ \ell_+ & -1 & \ell_- \end{pmatrix}, \text{ where } \ell_\pm := \frac{1}{2}\left(-1 \pm \sqrt{5}\right).$$

Type H^*: A subgroup of order 180 generated by **Type** H and

$$W := \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}.$$

Type I: Simple group of order 180 generated by T and

$$S_7 := \begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7^4 \end{pmatrix} \quad \text{and} \quad V := \frac{1}{\sqrt{-7}} \begin{pmatrix} \zeta_7^4 - \zeta_7^3 & \zeta_7^2 - \zeta_7^5 & \zeta_7 - \zeta_7^6 \\ \zeta_7^2 - \zeta_7^5 & \zeta_7 - \zeta_7^6 & \zeta_7^4 - \zeta_7^3 \\ \zeta_7 - \zeta_7^6 & \zeta_7^4 - \zeta_7^3 & \zeta_7^2 - \zeta_7^5 \end{pmatrix}.$$

Type I^*: Group of order 504 generated by **Type** I and W.

Type J: Group of order 1080 generated by **Type** H and

$$E_4 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\zeta_3 \\ 0 & -\zeta_3^2 & 0 \end{pmatrix}.$$

Any crepant resolution of $X = \mathbb{C}^3/G$ determines a triangulation of the junior 3-simplex Δ_3 into $|G|$ equal volume simplexes, following [CR02] we describe an algorithm for subdividing Δ_3 into appropriate equal area triangles.

Let $G \subseteq \mathrm{SL}_3(\mathbb{C})$ be a finite abelian subgroup. Denote by L the lattice containing $\mathbb{Z}^3$ and generated by all elements in G of the form $\frac{1}{r}(a_1, a_2, a_3)$. Let Δ_3 be the junior simplex in L with vertices

$$e_1 = \frac{1}{r}(r,0,0) = (1,0,0), \quad e_2 = \frac{1}{r}(0,r,0) = (0,1,0), \quad e_3 = \frac{1}{r}(0,0,r) = (0,0,1).$$

Let $\mathbb{R}^2_{\Delta_3}$ be the affine plane spanned by Δ_3 and $\mathbb{Z}^2_{\Delta_3} := L \cap \mathbb{R}^2_{\Delta_3}$ its affine lattice. Take each e_i as an origin.

For $i \in \{1,2,3\}$ (we use indices mod 3) consider the convex hull of the lattice points in $\Delta_3 \setminus \{e_i\}$:

$$f_{i,0} \qquad f_{i,1} \qquad \cdots \qquad f_{i,k_i+1}$$

primitive vector along the $\overline{e_i e_{i-1}}$ primitive vector along the $\overline{e_i e_{i+1}}$

Then

$$f_{i,j-1} + f_{i,j+1} = b_{i,j} f_{i,j} \quad \text{for } j = 1, 2, \dots, k_i,$$

hence

$$\mathbb{Z}^2 + \mathbb{Z} \cdot \frac{1}{r_i}(1, b_i) = \mathbb{Z}^2(f_{i,0}, f_{i,k_i+1}) + \mathbb{Z} \cdot f_{i,1},$$

where r_i, b_i be coprime integers defined as Hirzebruch-Jung continued fraction

$$\frac{r_i}{b_i} = [b_{i,1}, b_{i,2}, \dots, b_{i,k_i}].$$

According to previous section, drawing lines $\overline{e_i f_{i,j}}$ we get a fan of a crepant resolution of singularity $\frac{1}{r_i}(1, b_i)$.

Algorithm 2.1.22.

1. For any $i \in \{1, 2, 3\}$ and $j \in \{0, 1, \dots, k_i\}$ draw the line $L_{i,j}$ out of e_i equal to $\overline{e_i f_{ij}}$. Mark the label $b_{i,j}$ above $L_{i,j}$.

2. Extend $L_{i,j}$ by the following rule:

   - if two or more lines meet, the line with greater label ($b_{i,j}$) is extended with a new label $b_{i,j} - 1$,
   - if two or more lines meet with equal label, the process ends for these lines.

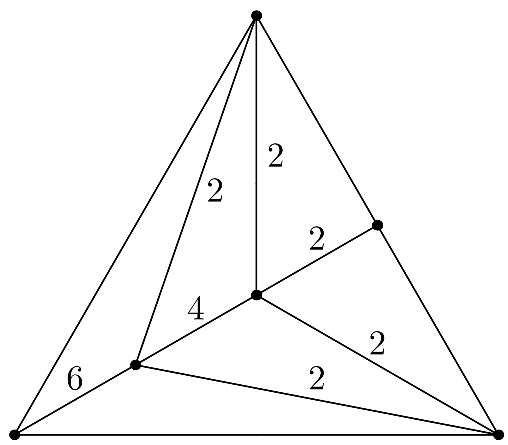
3. Continue the process from 2. as long as it is possible.

From the algorithm 2.1.22 we obtain triangulation of the junior simplex Δ_3 into *r-regular* triangles i.e. affine equivalent to the triangle with vertices $(0, 0)$, $(r, 0)$, $(0, r)$ for some $r \geq 1$, called the *side* of $\mathcal{T}$.

Finally we make so called *regular tessellation* of r-regular triangle i.e. subdivision of r-regular triangles (needed only if $r > 1$) into r^2 basic triangles with sides parallel to sides Δ_3.

Let us denote the final triangulation (toric fun) determined by tessellation of all r-regular triangles in Δ_3 by $\widetilde{C}$. In local coordinates the map from $\mathbb{C}^3$ to the resolution $X_{C',\widetilde{L}}$ of singularity $\frac{1}{r}(a_1, a_2, a_3)$ are given in affine charts by dual cone to any triangle in the triangulation $\widetilde{C}$.

Example 2.1.23. The analysis of the following example will be crucial in Chapter 3. Consider $\frac{1}{6}(1, 1, 4)$ singularity. One can check that points $\frac{1}{6}(1, 1, 4)$, $\frac{1}{6}(2, 2, 2)$ and $\frac{1}{6}(3, 3, 0)$ are the only junior points in Δ_3 (except vertices of the simplex). Therefore by the algorithm 2.1.22 we can add to Δ_3 the following data:



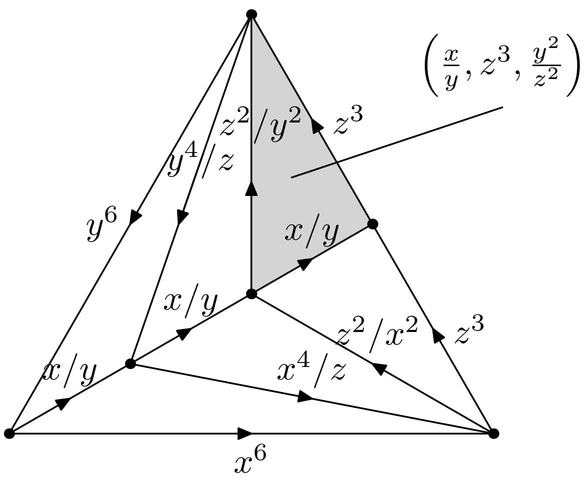
In fact $\frac{6}{4} = [\mathbf{2}, \mathbf{2}]$, $\frac{6}{1} = [\mathbf{6}]$ and

$$\begin{aligned}
\frac{1}{6}(6,0,0) + \frac{1}{6}(0,6,0) &= \mathbf{6} \cdot \frac{1}{6}(1,1,4),\\
\frac{1}{6}(0,0,6) + \frac{1}{6}(2,2,2) &= \mathbf{2} \cdot \frac{1}{6}(1,1,4),\\
\frac{1}{6}(1,1,4) + \frac{1}{6}(3,3,0) &= \mathbf{2} \cdot \frac{1}{6}(2,2,2),\\
\frac{1}{6}(3,3,0) + \frac{1}{6}(1,1,4) &= \mathbf{2} \cdot \frac{1}{6}(2,2,2),\\
\frac{1}{6}(2,2,2) + \frac{1}{6}(0,0,6) &= \mathbf{2} \cdot \frac{1}{6}(1,1,4).
\end{aligned}$$

Dual cone to any triangle in the above triangulation of Δ_3 (into 6 triangles) corresponds to affine chart in local coordinates of the map

$$\mathbb{C}^3 \to \mathbb{C}^3 \big/ _{\frac{1}{6}(1,1,4)}.$$

Since some rational function may corresponds to a common side of two triangles it is convenient to equipped this side with an arrow. Then we read rational functions in accordance with the agreed orientation of all triangles (including Δ_3), opposite arrow (to the fixed orientation) on the side indicates inverse of the rational attached to this side in the affine chart.



For example; consider grey triangle. Take duals in monomial ring to the columns of the following matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 2/6 & 2/6 & 2/6 \\ 3/6 & 3/6 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 2 \\ 0 & 3 & -2 \end{pmatrix}$$

obtaining the following rational functions functions $\frac{x}{y}$, z^3, $\frac{y^2}{z^2}$ opposite to the vertices $(1,0,0)$, $\frac{1}{6}(2,2,2)$, $\frac{1}{6}(3,3,0)$, respectively.

One can directly read that inverse map to the resolution are given in affine charts as

$$\left(x^6, \frac{z}{x^4}, \frac{y}{x}\right), \quad \left(\frac{x^4}{z}, \frac{z^2}{x^2}, \frac{y}{x}\right), \quad \left(\frac{x^2}{z^2}, z^3, \frac{y}{x}\right), \quad \left(\frac{x}{y}, \frac{z}{y^4}, y^6\right),$$
$$\left(\frac{x}{y}, \frac{z^2}{y^2}, \frac{y^4}{z}\right) \quad \text{or} \quad \left(\frac{x}{y}, z^3, \frac{y^2}{z^2}\right).$$

Dimension $d \geq 4$

The following example shows that in dimensions $d \geq 4$ the existence of any crepant resolution coming from toric resolution is delicate.

Example 2.1.24. Let $G = \{-\mathrm{id}, \mathrm{id}\} \subseteq \mathrm{Sl}_4(\mathbb{C})$. Then ${}^{\mathbb{C}^4}\!/_{G}$ does not have a toric crepant resolution of singularities.

Proof. There are no junior points in group $G \simeq \mathbb{Z}_2$, hence Δ_4 cannot be subdivided into 2 equal-volume polytopes spanned by junior points. □

Example 2.1.24 can be easily generalize to the following one:

Example 2.1.25. Let G be the order d cyclic group

$$G = \langle \zeta_d \cdot \mathrm{id} \rangle, \ 1 < d < n, \ d \mid n.$$

Then ${}^{\mathbb{C}^n}\!/_{G}$ has no toric crepant resolution.

In fact ${}^{\mathbb{C}^n}\!/_{G}$ has no crepant resolution, because any such resolution of singularities is automatically toric (see [Roa97]).

S. Davis in her PhD [Dav12] presented several approaches to the problem of existence of crepant resolution in 4 dimensional case. She gave also algorithm which determines, for quotients by cyclic subgroup of $\mathrm{Sl}_4(\mathbb{C})$ whether or not a crepant resolution exists.

It seems that junior points are crucial in the problem of existence of crepant resolution of a quotient ${}^{\mathbb{C}^n}\!/_{G}$:

Theorem 2.1.26 ([Yam18])**.** *Let G be a finite subgroup $G \subset \mathrm{SL}_n(\mathbb{C})$, then if ${}^{\mathbb{C}^n}\!/_{G}$ has a crepant resolution then G is generated by junior elements.*

In [DH97] D. I. Dais and M. Henk proved that 1-parameter quotient cyclic singularity

$$\frac{1}{r}\Big(r-(n-1), \underbrace{1, 1, \ldots, 1}_{n-1}\Big) \tag{2.1.2}$$

has (unique) crepant resolution iff $r \equiv 0 \pmod{n-1}$ or $r \equiv 1 \pmod{n-1}$.

Much more study has been done in order to find necessary and sufficient condition for cyclic (2-parameter) quotient singularities of the the form

$$\frac{1}{r}\Big(a, r-a-(n-2), \underbrace{1, 1, \dots, 1}_{n-2}\Big) \tag{2.1.3}$$

to have a crepant resolution.

Such condition has been given by D. I. Dais, U. U. Haus and M. Henk in [DHH98] and different one by S. Davis T. Logvinenko and M. Reid in [SR12] (see also project webpage: http://www.cantab.net/users/t.logvinenko/Traps/index.html).

Theorem 2.1.27 ([SR12])**.** *Consider* 2*-parameter cyclic singularity*

$$\frac{1}{r}(a, b, 1, 1, \dots, 1). \tag{2.1.4}$$

Then:

- *if* $\gcd(r, a, b) > 1$, *then* 2.1.4 *has a crepant resolution,*
- *if* $\gcd(r, a, b) = 1$ *and* $\gcd(r, a) > 1$ *or* $\gcd(r, b) > 1$, *then* 2.1.4 *has a crepant resolution iff*

  $$\frac{1}{r}(0, k_1, r_1, \dots, r_1) \quad \text{and} \quad \frac{1}{r}(k_2, 0, r_2, \dots, r_2)$$

  are junior points with

  $$r = r_1 \gcd(r, a) = r_2 \gcd(r, b) = k_1 + r_1(n-2),$$

- *if* $\gcd(r, a) = 1$ *i.e. we treat only the case*

  $$\frac{1}{r}(0, d, c, \dots, c),$$

  where $r = 1 + d + (n-2)c$. *In that case* 2.1.4 *has a crepant resolution iff the following two condition are satisfying:*

  - *the point nearest to the* $x_1 = 0$ *face of junior simplex is the junior point*

    $$\frac{1}{r}(0, d, c, \dots, c),$$

  - *the Hirzebruch-Jung continued fraction of* $\frac{r}{d}$ *has every entry congruent to* $2 \pmod{n-2}$.

In [DHZ06] D. I. Dais, M. Henk and G. M. Ziegler proved that for all $k \geq 2$ and $\ell \geq 3$ the following quotient singularity

$$\frac{1}{\left(\frac{k^{\ell}-1}{k-1}\right)}\left(1, k, k^2, \dots, k^{\ell-2}, k^{\ell-1}\right)$$

admits a crepant resolution.

Recently K. Sato and Y. Sato in [SS20] used generalization of Hirzebruch-Jung continued fractions introduced by T. Ashikaga in [Ash19] to find necessary and sufficient condition for the so-called *Fujiki-Oka resolutions* (see [Ash19], [Oka87]) of Gorenstein abelian quotient singularities to be crepant in all dimensions.

Definition 2.1.28. Let $r, n \geq 1$ be integers and let $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n$ with $0 \leq a_i \leq r-1$ for $1 \leq i \leq n$. An *n-dimensional proper function* is the following symbol

$$\frac{\mathbf{a}}{r} := \frac{(a_1, \dots, a_n)}{r}.$$

If at least one coordinate of $\mathbf{a}$ is 1, then $\frac{\mathbf{a}}{r}$ is called *semi- unimodular proper fraction*.

The *age* of an element $\dfrac{(a_1, \dots, a_n)}{r}$ is defined to be

$$\operatorname{age}\left(\frac{(a_1, \dots, a_n)}{r}\right) = \frac{1}{r}\sum_{i=1} a_i.$$

Definition 2.1.29. Let $\dfrac{\mathbf{a}}{r} = \dfrac{(1, a_2, \dots, a_n)}{r}$ be a semi-unimodular proper fraction. Then for $2 \leq i \leq n$, the *i-th remainder map* is defined as follows:

$$R_i\left(\frac{\mathbf{a}}{r}\right) = \begin{cases} \dfrac{(1, \overline{a_2}^{a_i}, \dots, \overline{a_{i-1}}^{a_i}, \overline{-r}^{a_i}, \overline{a_{i+1}}^{a_i}, \dots, \overline{a_n}^{a_i})}{a_i}, & \text{if } a_i \neq 0, \\ \infty & \text{if } a_i = 0, \end{cases}$$

where $\bar{x}^y$ denotes $x \pmod{y} \in \{0, 1, \dots, y-1\}$, we put also $R_i(\infty) = \infty$.

The *remainder polynomial* is a formal polynomial with variables $x_2, \dots, x_n$ defined by

$$R_*\left(\frac{\mathbf{a}}{r}\right) := \left(\frac{\mathbf{a}}{r}\right) + \sum_{\substack{(i_1, i_2, \dots, i_\ell) \in \{2,3,\dots,n\}^\ell, \\ \ell \geq 1}} (R_{i_\ell} \circ \dots \circ R_{i_2} \circ R_{i_1})\left(\frac{\mathbf{a}}{r}\right) \cdot x_{i_1} x_{i_2} \dots x_{i_\ell},$$

we exclude terms with coefficients ∞ or $\dfrac{(0, \dots, 0)}{1}$.

The following theorem gives sufficient condition for singularity of type $\dfrac{1}{r}(1, a_2, \dots, a_n)$ to have crepant resolution.

Theorem 2.1.30. *For a cyclic quotient singularity of type* $\frac{1}{r}(1, a_2, \ldots, a_n)$*, the Fujiki-Oka resolution is crepant if the ages of all the coefficients of the corresponding remainder polynomial* $R_*\left(\frac{(1, a_2, \ldots, a_n)}{r}\right)$ *are equal to* 1.

Moreover if $R_*\left(\frac{(1, a_2, \ldots, a_n)}{r}\right)$ *has iterated term (i.e.* $x_i x_i \ldots x_i$*) with coefficient at least* 2*, then singularity* $\frac{1}{r}(1, a_2, \ldots, a_n)$*, has no a crepant resolution.*

Example 2.1.31.

$$R_*\left(\frac{(1,2,3)}{6}\right) = \frac{(1,2,3)}{6} + \frac{(1,0,1)}{2}x_2 + \frac{(1,2,0)}{3}x_3 + \frac{(1,0,0)}{1}x_2x_3 + \frac{(1,1,0)}{2}x_3x_2 + \frac{(1,0,0)}{2}x_3x_2x_2,$$

and

$$\begin{aligned} \operatorname{age}\left(\frac{(1,2,3)}{6}\right) &= \operatorname{age}\left(\frac{(1,0,1)}{2}\right) = \operatorname{age}\left(\frac{(1,2,0)}{3}\right) = \\ &= \operatorname{age}\left(\frac{(1,0,0)}{1}\right) = \operatorname{age}\left(\frac{(1,1,0)}{2}\right) = \operatorname{age}\left(\frac{(1,0,0)}{2}\right) = 1. \end{aligned}$$

Remark 2.1.32. The above condition is not necessary: singularity $\frac{1}{24}(1, 5, 6, 12)$ does not satisfy 2.1.30 (coefficient in x_2x_4 is equal to $\frac{1}{2}(1, 1, 1, 1)$) but can be resolved in crepant way (see example 3.12 of [SS20].

In [Sat10] K. Sato found some infinite series of non-cyclic finite subgroups of $G \subseteq \mathrm{SL}_4(\mathbb{C})$ such that $\mathbb{C}^4/G$ has a toric projective crepant resolution of singularities.

Theorem 2.1.33 ([Sat10])**.** *Let* p *be a prime number and let* $G \subset \mathrm{SL}_4(\mathbb{C})$ *be one of the following group:*

1. $\left\langle \frac{1}{p}(1, 0, a, p-a-1), \frac{1}{p}(0, 1, b, p-b-1) \right\rangle \simeq \mathbb{Z}_p^2$,
2. $\left\langle \frac{1}{p}(1, a, 0, p-a-1), \frac{1}{p}(0, 0, 1, p-1) \right\rangle \simeq \mathbb{Z}_p^2$, *where* $a \neq 0$,
3. $\left\langle \frac{1}{p}(0, 1, 0, p-1), \frac{1}{p}(0, 0, 1, p-1) \right\rangle \simeq \mathbb{Z}_p^2$,
4. $\left\langle \frac{1}{p}(1, 0, 0, p-1), \frac{1}{p}(0, 1, b, p-1), \frac{1}{p}(0, 0, 1, p-1) \right\rangle \simeq \mathbb{Z}_p^3$.

Then $\mathbb{C}^4/G$ *admits crepant resolution in the case 4, 3 and 2 where* $a \in \left\{1, \frac{p-1}{2}, p-2, p-1\right\}$ *and 1 where* $a = b = 1$.

See also [SS20] for connections between crepant resolution of singularities of ${}^{\mathbb{C}^n}\!/_{G}$ when G can be decomposed to the cyclic components and Fujiki-Oka resolutions.

When the finite subgroup $G \subseteq \mathrm{SL}_n(\mathbb{C})$ is non-abelian, we cannot use toric geometry in order to find crepant resolution. However, if one can divide the group G into abelian subgroups, then we may obtain a crepant resolution as a combination of toric crepant resolutions of singularities.

Theorem 2.1.34 ([HIS17])**.** *Let $G \subset \mathrm{SL}_4(\mathbb{C})$ be one of the following group:*

- $\left\langle \frac{1}{n}(1,0,0,-1), \frac{1}{n}(0,1,0,-1), \frac{1}{n}(0,0,1,-1) \right\rangle$*, where $n \geq 2$,*
- $\left\langle \frac{1}{2n}(1,n-1,1,n-1), \frac{1}{2}(0,1,0,1) \right\rangle$*, where $n \geq 1$,*
- $\left\langle \frac{1}{4n}(1,2n-1,1,2n-1) \right\rangle$*, where $n \geq 2$,*
- $\left\langle \frac{1}{m}(1,-1,0,0), \frac{1}{n}(0,0,1,-1) \right\rangle$*, where $m \geq 3, n \geq 2$.*

Let $K \subset \mathrm{SL}_4(\mathbb{C})$ be a group generated by

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SL}_4(\mathbb{C}).$$

Consider a non-abelian finite subgroup $G = \langle H, K \rangle$ of $\mathrm{SL}_4(\mathbb{C})$, then ${}^{\mathbb{C}^4}\!/_{G}$ has a crepant resolution of singularities.

We and this section with a conjecture proposed by Reid ([Rei92]):

Conjucture 2.1.35 ([Rei92])**.** Let G be a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$ and let $\widetilde{X}$ be a crepant resolution of $X := {}^{\mathbb{C}^n}\!/_{G}$. Then there exists basis of $H^*(\widetilde{X}, \mathbb{Q})$ consisting of algebraic cycles in $1-1$ correspondence with conjugacy classes of G, such that conjugacy classes with age k correspond to basis elements of $H^{2k}(\widetilde{X}, \mathbb{Q})$.

In particular $b_{2k}(\widetilde{X})$ ($2k$-th Betti number) is the number of conjugacy classes G with age k, and $b_{2k+1}(\widetilde{X}) = 0$, so

$$e(\widetilde{X}) = \#\mathrm{Conj}(G) \; - \; \text{the number of conjugacy classes of } G.$$

§ 2.2 Chen-Ruan cohomology

In [CR04] W. Chen and Y. Ruan introduced new cohomology theory for orbifold. Let X be a projective variety and G be a finite group which acts on X.

Definition 2.2.1. For a variety ${}^{X}\!/_{G}$ define the *Chen-Ruan cohomology* by

$$H^{i,j}_{\text{orb}}\left({}^{X}\!/_{G}\right) := \bigoplus_{[g]\in \text{Conj}(G)} \left(\bigoplus_{U\in\Lambda(g)} H^{i-\text{age}(g),\, j-\text{age}(g)}(U) \right)^{\text{C}(g)}, \tag{2.2.1}$$

where $\text{Conj}(G)$ is the set of conjugacy classes of G (we choose a representative g of each conjugacy class), $\text{C}(g)$ is the centralizer of g, $\Lambda(g)$ denotes the set of irreducible connected components of the set fixed by $g \in G$ and $\text{age}(g)$ is the age of the matrix of linearised action of g near a point of U.

The dimension of $H^{i,j}_{\text{orb}}\left({}^{X}\!/_{G}\right)$ will be denoted by $h^{i,j}_{\text{orb}}\left({}^{X}\!/_{G}\right)$.

Remark 2.2.2. The definition makes sense since age is locally constant along each component of $\text{Fix}(g)$. The components of $\text{Fix}(g)$ are not necessarily invariant under the action of $\text{C}(g)$, so in 2.2.1 we need to consider the action of $C(g)$ on the inner direct sum (not on each summand separately).

If the group G is cyclic of a prime order p i.e $G = \langle\alpha\rangle$, then formula 2.2.1 simplifies to

$$H^{i,j}_{\text{orb}}\left({}^{X}\!/_{G}\right) = H^{i,j}(X)^G \oplus \bigoplus_{U\in\Lambda(\alpha)} \bigoplus_{k=1}^{p-1} H^{i-\text{age}\left(\alpha^k\right),\, j-\text{age}\left(\alpha^k\right)}(U).$$

An important feature of Chen-Ruan cohomology is the possibility of computing Hodge numbers of a crepant resolution of singularities of a quotient variety, without referring to an explicit construction of such a resolution.

Theorem 2.2.3 ([Yas04])**.** *Let G be a finite group acting on an algebraic smooth variety X. If there exists a crepant resolution $\widetilde{{}^{X}\!/_{G}}$ of variety ${}^{X}\!/_{G}$, then the following equality holds*

$$h^{i,j}\left(\widetilde{{}^{X}\!/_{G}}\right) = h^{i,j}_{\text{orb}}\left({}^{X}\!/_{G}\right).$$

For a systematic exposition of the orbifold Chen-Ruan cohomology see [ALR07].

§ 2.3 Hodge numbers of the finite quotient

Let X_i be a variety of Calabi-Yau type of dimension n_i with purely non-symplectic automorphism $\phi_{i,d} : X_i \to X_i$ of order d i.e. for any $\omega_{X_i} \in H^{n_i,0}(X_i, \mathcal{O}_{X_i})$ and the fixed d-th root of unity ζ_d:

$$\phi_{i,d}\omega_{X_i} = \zeta_d \omega_{X_i},$$

for $i = 1, 2, \dots, n$. Consider the following group

$$G_{d,n} := \{m := (m_1, m_2, \dots, m_n) \in \mathbb{Z}_d^n : m_1 + m_2 + \dots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

which acts symplecticly on $X_1 \times X_2 \times \dots \times X_n$ by $\phi_{i,d}^{m_i}$ on the i-th factor. Suppose that there exists a crepant resolution $\widetilde{\mathcal{X}_{d,n}}$ of the quotient variety

$$\mathcal{X}_{d,n} := X_1 \times X_2 \times \dots \times X_n \big/ \mathbb{Z}_d^{n-1},$$

then $\mathcal{X}_{d,n}$ is of Calabi-Yau type. The aim of this section is to give a formula for the Hodge number of $\mathcal{X}_{d,n}$ using Chen-Ruan cohomology. We shall use the following obvious lemma:

Lemma 2.3.1. *Let $V_1, V_2, \dots, V_n$ be vector spaces over field of k containing ζ_d. Assume that $\alpha_i \in \mathrm{End}(V_i)$ for $i = 1, 2, \dots n$ is an automorphism of order d. Then the fixed locus of $G_{d,n}$ acting on $V_1 \times V_2 \times \dots \times V_n$ by*

$$\alpha_g : G_{d,n} \ni g \mapsto \alpha_1^{g_1} \times \dots \times \alpha_n^{g_n} \in \mathrm{End}(V_1 \times V_2 \times \dots \times V_n)$$

equals

$$\bigoplus_{m=0}^{d-1} \left(V_1\right)_{\zeta_d^m} \otimes \dots \otimes \left(V_n\right)_{\zeta_d^m},$$

where $\left(V_i\right)_{\zeta_d^m}$ denotes m-th eigenspace of $\mathbb{Z}_d$ action on V_i, for $i \in \{1, 2, \dots, n\}$ and $m \in \{0, 1, \dots, d-1\}$.

Proof. Since the action of $G_{d,n}$ is diagonalizable we can consider only the tensor product of eigenvectors $v_1 \otimes \dots \otimes v_n$ where $\alpha_i(v_i) = \zeta_d^{m_i} v_i$. If $m_i \neq m_j$ for some $1 \le i < j \le n$ then consider an element

$$g_{ij} = \big(0, \dots, \underbrace{1}_{i-\text{th place}}, \dots \underbrace{d-1}_{j-\text{th place}}, \dots 1\big).$$

Since we have

$$\alpha_{g_{ij}}(v_1 \otimes \dots \otimes v_n) = \zeta_d^{m_i + (d-1)m_j} v_1 \otimes \dots \otimes v_n$$

the vector $v_1 \otimes \dots \otimes v_n$ is fixed by $\alpha_{g_{ij}}$ iff $d \mid m_i + (d-1)m_j$ which is equivalent to $m_i = m_j$. $\square$

Observe first that for $i \in \{1, 2, \dots, n\}$, $0 \leq m_i \leq d-1$, the automorphism $\phi_{i,d}^{m_i}$ has local diagonalization near a point of $\operatorname{Fix}\left(\phi_{i,d}^{m_i}\right)$ of the following form

$$\left(\zeta_d^{\alpha_1 m_i}, \dots, \zeta_d^{\alpha_{n_i} m_i}\right),$$

where $\zeta_d^{\alpha_1} \cdot \ldots \cdot \zeta_d^{\alpha_{n_i}} = \zeta_d$. Consequently

$$\operatorname{age}\left(\phi_{i,d}^{m_i}\right) = \frac{m_i}{d} + \lambda_i,$$

where λ_i is a non-negative integer, moreover λ_i is constant along every component of $\operatorname{Fix}(\alpha_i)$.

Definition 2.3.2. Define

$$X_{i,m_i,\lambda_i} = \left\{ x \in \operatorname{Fix}\left(\phi_{i,d}^{m_i}\right) \,:\, \operatorname{age}\left(\phi_{i,d}^{m_i}\right) = \frac{m_i}{d} + \lambda_i \ \text{ near } x \right\}$$

and

$$F_{X_{i,m_i,j}}(X,Y) = \sum_{\lambda_i \geq 0} \sum_{0 \leq p,q \leq \dim X_i} \dim_{\mathbb{C}} \left(H^{p,q}\left(X_{i,m_i,\lambda_i}\right)_{\zeta_d^j} \right) X^{p+\lambda_i} Y^{q+\lambda_i}.$$

Theorem 2.3.3. *Under the above assumptions*

$$h^{p,q}\left(\widetilde{\mathcal{X}_{d,n}}\right) = \sum_{j=0}^{d-1} \prod_{i=1}^{n} \left(\sum_{m=0}^{d-1} \sqrt[d]{(XY)^m} \cdot F_{X_{i,m,j}}(X,Y) \right) [X^p Y^q], \tag{2.3.1}$$

where $\mathcal{P}[X^pY^q]$ denotes the coefficients of polynomial $\mathcal{P}$ in X^pY^q.

Proof. By the Yasuda Theorem 2.2.3

$$H^{p,q}\left(\widetilde{\mathcal{X}_{d,n}}\right) = H^{p,q}_{\text{orb}}\left(\mathcal{X}_{d,n}\right).$$

Now by the Künneth formula for Hodge groups it follows that

$$H^{p,q}\left(\widetilde{\mathcal{X}_{d,n}}\right) = \sum_{m \in G_{n,d}} \left(\sum_{\lambda \in \mathbb{N}^n} \sum_{\substack{\mathfrak{p},\mathfrak{q} \in \mathbb{N}^n \\ |\mathfrak{p}| = p - \frac{1}{d}|\mathfrak{p}| - |\lambda| \\ |\mathfrak{q}| = q - \frac{1}{d}|\mathfrak{q}| - |\lambda|}} H^{p_1,q_1}\left(X_{1,m_1,\lambda_1}\right) \otimes \ldots \otimes H^{p_n,q_n}\left(X_{n,m_n,\lambda_n}\right) \right)^{G_{n,d}},$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\mathfrak{p} = (p_1, p_2, \dots, p_n)$, $\mathfrak{q} = (q_1, q_2, \dots, q_n)$ and $|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$, $|\mathfrak{p}| = p_1 + p_2 + \ldots + p_n$, $|\mathfrak{q}| = q_1 + q_2 + \ldots + q_n$.

Since $\phi_{d,n}\left(X_{i,m_i,\lambda_i}\right) = X_{i,m_i,\lambda_i}$, the group $G_{n,d}$ acts separately on each summand of the inner sum, hence by the lemma 2.3.1 we get

$$H^{p,q}\left(\widetilde{\mathcal{X}_{d,n}}\right) = \sum_{m \in G_{n,d}} \sum_{\lambda \in \mathbb{N}^n} \sum_{\substack{\mathfrak{p},\mathfrak{q} \in \mathbb{N}^n \\ |\mathfrak{p}| = p - \frac{1}{d}|\mathfrak{p}| - |\lambda| \\ |\mathfrak{q}| = q - \frac{1}{d}|\mathfrak{q}| - |\lambda|}} \sum_{j=0}^{d} H^{p_1,q_1}(X_{1,m_1,\lambda_1})_{\zeta_d^j} \otimes \ldots \otimes H^{p_n,q_n}(X_{n,m_n,\lambda_n})_{\zeta_d^j}.$$

Taking dimensions and forming the generating function we get

$$\sum_{0\le p,q\le \dim \widetilde{\mathcal{X}_{d,n}}} h^{p,q}_{\mathrm{orb}}\left(\widetilde{\mathcal{X}_{d,n}}\right) X^pY^q = \sum_{0\le p,q\le \dim \widetilde{\mathcal{X}_{d,n}}} \sum_{m\in G_{n,d}} \sum_{\lambda\in\mathbb{N}^n} \sum_{j=0}^{d} \sum_{\substack{\mathfrak{p},\mathfrak{q}\in\mathbb{N}^n \\ |\mathfrak{p}|=p \\ |\mathfrak{q}|=q}} h^{p_1-\lambda_1,q_1-\lambda_1}\left(X_{1,m_1,\lambda_1}\right)_{\zeta_d^j}\cdot\ldots\cdot$$

$$\cdot\, h^{p_n-\lambda_n,q_n-\lambda_n}\left(X_{n,m_n,\lambda_n}\right)_{\zeta_d^j} X^pY^q\cdot (XY)^{\frac{m_1+\ldots+m_n}{d}} =$$

$$= \sum_{m\in G_{n,d}} \sum_{\lambda\in\mathbb{N}^n} \sum_{j=0}^{d}\left(\sum_{p_1,q_1\ge 0} h^{p_1-\lambda_1,q_1-\lambda_1}\left(X_{1,m_1,\lambda_1}\right)_{\zeta_d^j} X^{p_1}Y^{q_1}\cdot (XY)^{\frac{m_1}{d}}\right)\times\ldots\times$$

$$\times\left(\sum_{p_n,q_n\ge 0} h^{p_n-\lambda_n,q_n-\lambda_n}\left(X_{n,m_n,\lambda_n}\right)_{\zeta_d^j} X^{p_n}Y^{q_n}\cdot (XY)^{\frac{m_n}{d}}\right) =$$

$$= \sum_{m\in G_{n,d}} \sum_{\lambda\in\mathbb{N}^n} \sum_{j=0}^{d-1}\left(\sum_{p_1,q_1\ge 0} h^{p_1,q_1}\left(X_{1,m_1,\lambda_1}\right)_{\zeta_d^j} X^{p_1}Y^{q_1}\right)(XY)^{\lambda_1+\frac{m_1}{d}}\cdot\ldots\cdot$$

$$\cdot\left(\sum_{p_n,q_n\ge 0} h^{p_n,q_n}\left(X_{n,m_n,\lambda_n}\right)_{\zeta_d^j} X^{p_n}Y^{q_n}\right)(XY)^{\lambda_n+\frac{m_n}{d}}.$$

Enlarging the exterior sum $\sum\limits_{m\in G_{n,d}}(\ldots)$ to $\sum\limits_{m\in\mathbb{Z}_d^n}(\ldots)$ we introduce only terms with fractional powers of X and Y, hence $h^{p,q}\left(\widetilde{\mathcal{X}_{d,n}}\right)$ is the coefficient in X^pY^q of the following Puiseux polynomial:

$$\sum_{m\in\mathbb{Z}_d^n} \sum_{\boldsymbol{\lambda}\in\mathbb{N}^n} \sum_{j=0}^{d-1}\left(\sum_{p_1,q_1\ge 0} h^{p_1,q_1}\left(X_{1,m_1,\lambda_1}\right)_{\zeta_d^j} X^{p_1}Y^{q_1}\right)(XY)^{\lambda_1+\frac{m_1}{d}}\cdot\ldots\cdot$$

$$\cdot\left(\sum_{p_n,q_n\ge 0} h^{p_n,q_n}\left(X_{n,m_n,\lambda_n}\right)_{\zeta_d^j} X^{p_n}Y^{q_n}\right)(XY)^{\lambda_n+\frac{m_n}{d}} =$$

$$= \sum_{j=0}^{d-1}\prod_{i=1}^{n}\sum_{m=0}^{d-1}\sqrt[d]{(XY)^m}\cdot F_{X_i,m_i,j}(X,Y).$$

□

Therefore, in order to compute Poincaré polynomial of $\widetilde{\mathcal{X}_{d,n}}$ for any $i\in\{1,2,\ldots,n\}$ it is enough to produce the following tables

$k \backslash j$	0	1	$\cdots$	j	$\cdots$	$d-1$
0	$F_{X_i,0,0}$	$F_{X_i,0,1}$		$F_{X_i,0,j}$		$F_{X_i,0,d-1}$
1	$F_{X_i,1,0}$	$F_{X_i,1,1}$		$F_{X_i,1,j}$		$F_{X_i,1,d-1}$
2	$F_{X_i,2,0}$	$F_{X_i,2,1}$	$\cdots$	$F_{X_i,2,j}$	$\cdots$	$F_{X_i,2,d-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$d-1$	$F_{X_i,d-1,0}$	$F_{X_i,d-1,1}$		$F_{X_i,d-1,j}$		$F_{X_i,d-1,d-1}$
				$=: v_{X_i,j}$		

Table 2.1: $F_{X_i,j,k}$

for any considered variety X_i. Then we compute scalar product of vectors $v_{X_i,j}$ and

$$v_d := \left(1, \sqrt[d]{(XY)}, \sqrt[d]{(XY)^2}, \dots, \sqrt[d]{(XY)^{d-1}}\right)$$

for $1 \leq j \leq n$. Finally we multiply all values of $v_{X_i,j} \circ v_d$ for $j \in \{1, 2, \dots, n\}$ and add all products for $i \in \{1, 2, \dots, n\}$.

§ 2.4 Stringy Euler Characteristic

One of the most important cohomological invariant of the finite quotient of compact manifolds conjectured by "physicists" is *physicists (stringy) Euler characteristic*.

Let G be a finite group acting on a compact, smooth differentiable manifold X. For any $g \in G$ let

$$X^g := \{x \in X : g(x) = x\}.$$

In [Dix+85; Dix+86] L. Dixon, J. Harvey, C. Vafa and E. Witten proposed the following *orbifold Euler number*:

$$e_{\text{orb}}\left({}^{X}\!/_{G}\right) := \frac{1}{\#G} \sum_{\substack{(g,h) \in G \times G \\ gh = hg}} e(X^g \cap X^h). \tag{2.4.1}$$

According to [HH90] the formula 2.4.1 can be rewritten as

$$e_{\text{orb}}\left({}^{X}\!/_{G}\right) := \sum_{[g] \in \mathrm{Conj}(G)} e\left({}^{X^g}\!/_{\mathrm{C}(g)}\right), \tag{2.4.2}$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G (we choose a representative g of each conjugacy class) and $\mathrm{C}(g)$ is the centralizer of g.

It is expected that orbifold Euler characteristic $e_{\text{orb}}\left({}^{X}\!/_{G}\right)$ coincides with the topological Euler characteristic $e\left(\widetilde{{}^{X}\!/_{G}}\right)$ of any crepant resolution of ${}^{X}\!/_{G}$. In fact:

Theorem 2.4.1 ([Roa89] (abelian case), [Bat99])**.** *Let $G \subset \mathrm{SL}_n(\mathbb{C})$ be a finite group acting on smooth algebraic variety X. If there exists a crepant resolution $\widetilde{{}^{X}\!/_{G}}$ of variety X/G, then the following equality holds*

$$e\left(\widetilde{{}^{X}\!/_{G}}\right) = e_{\text{orb}}\left({}^{X}\!/_{G}\right).$$

Some explicit examples of a group G and possible calculations were studied by T. Höfer and F. Hirzebruch in [HH90].

We shall use orbifold Euler characteristic in order to find relations between parameters attached to $K3$ surfaces in Chapter 5.

§ 2.5 Topological and holomorphic Lefchetz's numbers

In the following section we review basic information about topological and holomorphic Lefschetz numbers. We refer to [GH94] and [Pet86].

Let X be a compact oriented manifold and $f : X \to X$ any continuous map. The intersection number of the graph Γ_f of f and diagonal Δ_X in $X \times X$ is equal to the so called *topological Lefschetz number* given by the following formula:

$$\mathcal{L}_{\text{top}}(f) := \sum_{q \geq 0} (-1)^q \operatorname{tr}\left(f^* : H^q(X) \to H^q(X)\right).$$

Assuming X is a complex manifold and f a holomorphic map then intersection $\Gamma_f \cdot \Delta_X$ may be computed in the case when connected components C of the $\mathrm{Fix}(f)$ are non-degenerate i.e. C is a manifold and for any $P \in C$ the linear map $(\mathrm{id} - \mathrm{d}f_P)$ is invertible.

Theorem 2.5.1 ([Uen76])**.** *Let X be a compact complex manifold and $f : X \to X$ a holomorphic map with non-degenerate fixed locus then*

$$\mathcal{L}_{\text{top}}(f) = \sum_{C \in \mathrm{Fix}(f)} e(C), \tag{2.5.1}$$

where the sum is taken over connected components of $\mathrm{Fix}(f)$.

Formula 2.5.1 holds in particular when $f^n = \mathrm{id}$ for some $n \geq 1$ (see [Car57]).

The map f acts not only on the de Rham cohomology of X but on the Dolbeault cohomology too. Therefore there is a hope that the action of f on $H^{*,*}(X)$ will be reflected in local properties of f around the fixed point locus.

Definition 2.5.2. Let X be a compact complex manifold and $f : X \to X$ a holomorphic map. The number

$$\mathcal{L}_{\mathrm{hol}}(f) := \sum_{q \geq 0} (-1)^q \operatorname{tr}\left(f^* | H^{0,q}(X)\right)$$

is called *holomorphic Lefschetz number*.

According to [AS68b] and [AS68a] the holomorphic Lefschetz number can be computed in different way. For our purpose assume that X has dimension 2 and let G be a finite group of automorphisms of X.

Theorem 2.5.3 ([AS68b], [AS68a])**.** *For any $g \in G$ the following formula holds*

$$\mathcal{L}_{\mathrm{hol}}(g) = \sum_{j \in J} a(P_j) + \sum_{k \in K} b(C_k),$$

where sets $\{P_j\}_{j \in J}$ and $\{C_k\}_{k \in K}$ denote fixed points and fixed curves in $\mathrm{Fix}(g)$ *and*

$$a(P) := \frac{1}{\det(1 - g|\mathcal{T}_g)}, \quad \text{where } \mathcal{T}_P \text{ is a tangent space at } P,$$

$$b(C) := \frac{1 - g(C)}{1 - \zeta} - \frac{\zeta C^2}{(1 - \zeta)^2}, \quad \text{where } \zeta \text{ is an eigenvalue of } g \text{ on the normal bundle of } \mathrm{Fix}(g).$$

§ 2.6 Zeta function of an algebraic variety

Definition 2.6.1. Let $q = p^k$ be a prime power and $X/\mathbb{F}_q$ a variety defined over $\mathbb{F}_q$. Then *the zeta function* of $X/\mathbb{F}_q$ is defined by

$$Z_q(t) := \exp\left(\sum_{r=1}^{\infty} N_{q^r} \cdot \frac{t^r}{r}\right) \in \mathbb{Q}[[t]],$$

where N_{q^r} is the number of $\mathbb{F}_q$-rational points of X.

Let X be a d-dimensional Calabi-Yau manifold defined over $\mathbb{Q}$. For a given prime power q, denote by Frob_q the *Frobenius morphism* i.e.

$$\mathrm{Frob}_q(x_1, x_2, \dots, x_n) = \left(x_1^q, x_2^q, \dots, x_n^q\right) \quad \text{for any } (x_1, x_2, \dots, x_n) \in \overline{X_q}.$$

In general it is a very deep problem to compute the zeta function of a given variety. A. Weil in [Wei49] formulated a series of remarkable conjectures concerning zeta functions; they are called the *Weil Conjectures* although since 1974 they are theorems.

Theorem 2.6.2 (Weil Conjectures)**.** *Let X be a smooth d-dimensional projective variety defined over the field $\mathbb{F}_q$, where $q = p^k$ is a prime power. Then the function $Z_q(t)$ satisfies the following properties:*

***Rationality**: The zeta function is rational i.e.*

$$Z_q(t) = \frac{P_q(t)}{Q_q(t)} \quad \text{for some polynomials } P_q, Q_q \in \mathbb{Q}[t].$$

***Functional equation**: The function $Z_q(t)$ satisfies a functional equation:*

$$Z_q\left(\frac{1}{q^d t}\right) = \pm q^{\frac{de}{2}} t^e Z_q(t),$$

where e is the Euler characteristic of the variety X over $\mathbb{C}$.

***Analogon of the Riemann hypothesis**:*

$$Z_q(t) = \frac{P_{1,q}(t) \cdot P_{3,q}(t) \cdot \ldots \cdot P_{2d-1,q}(t)}{P_{0,q}(t) \cdot P_{2,q}(t) \cdot \ldots \cdot P_{2d,q}(t)},$$

where $P_{0,q}(t) = 1 - t$, $P_{2d,q}(t) = 1 - q^d t$ and

$$P_{i,q}(t) = \prod_{j=1}^{b_i} (1 - \alpha_{i,j} t)$$

for $1 \leq i \leq 2d-1$, where the $\alpha_{i,j}$ are algebraic integers of complex absolute value $|\alpha_{i,j}| = q^{1/2}$.

The Lefschetz fixed point formula applied to Frobenius morphism yields

$$N_q = \sum_{i=0}^{2d} (-1)^i \operatorname{tr}\left(\mathrm{Frob}_q^* \mid H^i_{\text{ét}}(X, \mathbb{Q}_l)\right).$$

We define polynomials

$$P_{i,q}(t) := \det\left(1 - \mathrm{Frob}_q^* t \mid H^i_{\text{ét}}(X, \mathbb{Q}_l)\right). \tag{2.6.1}$$

From here one can easily deduce that

$$Z_q(t) = \frac{P_{1,q}(t) \cdot P_{3,q}(t) \cdot \ldots \cdot P_{2d-1,q}(t)}{P_{0,q}(t) \cdot P_{2,q}(t) \cdot \ldots \cdot P_{2d,q}(t)},$$

in particular polynomials from 2.6.1 coincide with polynomials defined in Riemann hypothesis (iii) in 2.6.2.

Now we define i-th L series of X by the following rule:

$$L_i\left(H^i_{\text{ét}}\left(\overline{X}, \mathbb{Q}_l\right), s\right) := (*) \prod_{p \in \mathcal{P}} \frac{1}{P_{i,q}(q^{-s})},$$

where $\mathcal{P}$ is the set of primes of good reduction and $(*)$ denotes suitable Euler factors for the primes of bad reduction. Usually, the d-th L series $L\left(H^d_{\text{ét}}(\overline{X}, \mathbb{Q}_l), s\right)$ we denote by $L(X, s)$.

Definition 2.6.3. The Hasse-Weil zeta function of X is defined by

$$\xi(X, s) = \frac{\prod_{i=0}^{d} L_{2i-1}(X, s)}{\prod_{i=1}^{d} L_{2i}(X, s)}.$$

We denote by

$$\xi(\mathbb{Q}, s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$$

the Riemann zeta function.

2.6.1 Examples

Elliptic curve

Let $q = p^n$ be any prime power. The zeta function of elliptic curve E over $\mathbb{F}_q$ has the following form:

$$Z_q(E/\mathbb{F}_q)(T) = \frac{1 - a_q T + qT^2}{(1-T)(1-qT)},$$

where

$$a_q = q + 1 - \#E(\mathbb{F}_q) \quad \text{and} \quad |a_q| \le 2\sqrt{q}.$$

By the famous result of A. Wiles ([Wil95]) there is a cusp form

$$f(\tau) = \sum_{k=1}^{\infty} b_k \tau^k,$$

where $\tau := e^{2\pi i z}$ of weight 2 and level N such that

$$L(E, s) = L(f, s).$$

Consequently $a_p := b_p$ for all p of good reduction for E.

Moreover the Hasse-Weil zeta function of X is given by

$$\xi(E, s) = \frac{L_1(E, s)}{\xi(\mathbb{Q}, s)\xi(\mathbb{Q}, s-1)}.$$

K3 surface

The zeta function of $K3$ surface S over $\mathbb{F}_q$ has the following form:

$$Z_q(S/\mathbb{F}_q)(T) = \frac{1}{(1-T)P_{22}(T)(1-q^2T)},$$

where $P_{22}(T)$ is a polynomial of degree 22.

Taking decomposition of lattices

$$H^2(S,\mathbb{Z}) = NS(S) \oplus T(S),$$

where $NS(S)$ is a Neron-Severi lattice (torsion free lattice of rank $\rho(S) \leq 20$) and $T(S)$ is a transcendental part of $H^2(S,\mathbb{Z})$ (for a detailed exposition of $K3$ surface theory see [Huy16]), induces factorisation of the L-series of S i.e.

$$L(S,s) = L(NS(S)\otimes \mathbb{Q}_\ell, s) \cdot L(T(S)\otimes \mathbb{Q}_\ell, s)$$

and therefore

$$\xi(S,s) = \frac{1}{\xi(\mathbb{Q},s)L(S,s)\xi(\mathbb{Q},s-2)}.$$

Y. Goto, R. Livné, N. Yui in [GLY13] described 86 out of 95 families (from Reid's list [Rei80]) of $K3$ surfaces S over $\mathbb{Q}$ of CM type and proved the following theorem

Theorem 2.6.4 ([GLY13]). *For all* 86 *pairs* (S,σ) *of* $K3$ *surfaces with involution, there exists quadruple* $(\rho,\mathbb{K},\iota,\chi)$ *with the following properties:*

- ρ *is an (Artin) Galois representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *and the degree of* ρ *is equal to Picard number of* S,
- $\mathbb{K}$ *is a* CM *abelian extension of* $\mathbb{Q}$,
- $\iota : \mathbb{K} \to \mathbb{C}$ *is an embedding,*
- χ *is a Hecke character of* $\mathbb{K}$ *of* ∞ *type* $z \to \iota(z)^2$,

such that

$$\xi(S,s) = \frac{1}{\xi(\mathbb{Q},s)\xi(\mathbb{Q},s-2)L(\rho,s-1)L(\chi,s)}.$$

We refer to [HKS06], [Mey05] for more complete exposition of *modularity* of Calabi-Yau manifolds.

§ 2.7 Zeta functions of a finite quotient

To compute the zeta function of $X_{n,d}$ we describe a much more general approach. Firstly let us recall some basic notation from linear algebra.

Let V, W be finite dimensional vector spaces over a field $\mathbb{K}$ and take $L \in \mathrm{End}(V)$, $M \in \mathrm{End}(W)$. We define their characteristic polynomials:

$$f(t) := \det(\mathbb{1} - t \cdot L) \quad \text{and} \quad g(t) := \det(\mathbb{1} - t \cdot M).$$

Over an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ we have factorisations

$$f(t) = (1 - \lambda_1 t)(1 - \lambda_2 t) \dots (1 - \lambda_{\dim V} t)$$

and

$$g(t) = (1 - \mu_1 t)(1 - \mu_2 t) \dots (1 - \mu_{\dim W} t),$$

for some $\lambda_1, \lambda_2, \dots, \lambda_{\dim V}, \mu_1, \mu_2, \dots, \mu_{\dim W} \in \overline{\mathbb{K}}$ which are in fact eigenvalues of endomorphisms $L_{\overline{\mathbb{K}}}$ and $M_{\overline{\mathbb{K}}}$ respectively.

Denoting by $f \otimes g$ the characteristic polynomial of $L \otimes M$:

$$f \otimes g(t) = \det(\mathbb{1} - t \cdot L \otimes M),$$

can be written as:

$$f \otimes g(t) := \prod_{i=1}^{\dim V} \prod_{j=1}^{\dim W} (1 - \lambda_i \mu_j t).$$

One can see that

$$f \otimes g(t) = \prod_{i=1}^{\dim V} g(\lambda_i t) = \prod_{j=1}^{\dim W} f(\mu_j t).$$

Moreover one can check that the this polynomial can be computed using resultant i.e.

$$f \otimes g(t) = \mathrm{res}_s \left(f(s), s^{\deg(g)} \cdot g\left(\frac{t}{s}\right) \right).$$

The tensor product of polynomials extends uniquely to the case of rational functions $f = \frac{a}{b}$, $g = \frac{c}{d}$, where $a, b, c, d \in \mathbb{K}[t]$ by taking:

$$\frac{a}{b} \otimes \frac{c}{d} = \frac{(a \otimes c) \cdot (b \otimes d)}{(a \otimes d) \cdot (b \otimes c)}.$$

Rosen in [Ros07] introduced the *orbifold Frobenius morphisms* on the Chen-Ruan orbifold cohomology and use it define *orbifold zeta function*

$$Z_{\mathrm{CR}}(\mathcal{X},t) := \det\left(1 - \mathrm{Frob}_{\mathrm{orb}}\, t \mid H^*_{\mathrm{CR}}\left(\mathcal{X} \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_l\right)\right), \tag{2.7.1}$$

where $H^*_{\mathrm{CR}}(\mathcal{X}, \mathbb{Q}_l)$ is ℓ-adic Chen Ruan cohomology and $\mathrm{Frob}_{\mathrm{orb}}$ is the *orbifold Frobenius morphism* defined in Section 3 and 5, respectively of [Ros07]. It was also proven (Corollary 6.4) that for a crepant resolution $\widetilde{X} \to X$ of the course moduli scheme X of $\mathcal{X}$ the orbifold cohomological zeta function of $\mathcal{X}$ coincide with classical zeta function of $\widetilde{X}$ i.e.

$$Z_{H^*_{\mathrm{CR}}}(\mathcal{X},t) = Z_q(\widetilde{X},t).$$

Let X_i be a variety of Calabi-Yau type with automorphism $\phi_{i,d} : X_i \to X_i$ of order d such that $\phi^*_{i,d}(\omega_{X_i}) = \zeta_d \omega_{X_i}$ for $i = 1, 2, \ldots, n$. Consider the following group

$$G_{d,n} := \{m := (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n_d : m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}^{n-1}_d$$

which acts on $X_1 \times X_2 \times \ldots \times X_n$ by $\phi^{m_i}_{i,d}$ on the i-th factor. Suppose that there exists a crepant resolution $\widetilde{\mathcal{X}_{d,n}}$ of the quotient variety

$$\mathcal{X}_{d,n} := X_1 \times X_2 \times \ldots \times X_n \big/ \mathbb{Z}^{n-1}_d .$$

Theorem 2.7.1. *The Zeta function $Z_q\left(\widetilde{\mathcal{X}_{d,n}}\right)(T)$ equals the product of factors of the rational function in T*

$$\left(\prod_{j=0}^{d-1} \bigotimes_{i=1}^{n} \left(\prod_{m=0}^{d-1} Z_{X_i,m,j}\left(q^{\frac{m}{d}} T\right)\right)\right)^{(-1)^{n+1}} \tag{2.7.2}$$

which contain only integral powers of q, where

$$Z_{X_i,m,j}(T) = \prod_{\lambda_i \geq 0} \prod_{0 \leq k_i \leq 2\dim X_i} \det\left(1 - \mathrm{Frob}^*_q T \mid H^{k_i}\left(X_{i,m,\lambda_i}\right)_{\zeta^j_d}\right)^{(-1)^{k_i+1}} (q^{\lambda_i} T).$$

Proof. Similarly as in the proof of 2.3.3 merging formulas 2.7.1 and 2.2.1 we have the following formula for zeta function of $\widetilde{\mathcal{X}_{d,n}}$:

$$Z_q\left(\widetilde{\mathcal{X}_{d,n}}\right)(T) = \prod_{m \in G_{d,n}} \prod_{\lambda \geq 0} \prod_{|\mathfrak{k}| = k - 2\frac{|\mathrm{m}|}{d} - \lambda} \prod_{j=0}^{d-1} \bigotimes_{i=1}^{n} \det\left(1 - \mathrm{Frob}_q T \mid H^{k_i}\left(X_{i,m_i,\lambda_i}\right)\right)^{(-1)^{k_i+1}} =$$

$$= \prod_{m \in G_{d,n}} \prod_{\lambda \geq 0} \prod_{|\mathfrak{k}| = k} \prod_{j=0}^{d-1} \bigotimes_{i=1}^{n} \det\left(1 - \mathrm{Frob}_q T \mid H^{k_i}\left(X_{i,m_i,\lambda_i}\right)\right)^{(-1)^{k_i+1}} \left(q^{\frac{m_i}{d} + \lambda_i} T\right) =$$

$$= \prod_{m\in G_{d,n}} \prod_{\lambda\geq 0} \prod_{|\mathfrak{k}|=k} \prod_{j=0}^{d-1} \left(\bigotimes_{i=1}^{n} \det\left(1 - \mathrm{Frob}_q T \mid H^{k_i}\left(X_{i,m_i,\lambda_i}\right)\right) \right)^{(-1)^{|\mathfrak{k}|+n}} \left(q^{\frac{m_i}{d}+\lambda_i} T\right),$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, $\mathfrak{k} = (k_1, k_2, \dots, k_n)$, $\mathfrak{m} = (m_1, m_2, \dots, m_n)$ and $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$, $|\mathfrak{k}| = k_1 + k_2 + \dots + k_n$, $|\mathfrak{m}| = m_1 + m_2 + \dots + m_n$.

Extending the exterior product $\prod_{m\in G_{n,d}} (\dots)$ to $\prod_{m\in \mathbb{Z}_d^n} (\dots)$ we introduce only factors containing fractional powers of q, denote the resulting rational function by

$$W(T) = \prod_{m\in \mathbb{Z}_d^n} \prod_{\lambda\geq 0} \prod_{|\mathfrak{k}|=k} \prod_{j=0}^{d-1} \left(\bigotimes_{i=1}^{n} \det\left(1 - \mathrm{Frob}_q T \mid H^{k_i}\left(X_{i,m_i,\lambda_i}\right)\right) \right)^{(-1)^{|\mathfrak{k}|+n}} \left(q^{\frac{m_i}{d}+\lambda_i} T\right) =$$

$$= \left(\prod_{j=0}^{d-1} \bigotimes_{i=1}^{n} \left(\prod_{m=0}^{d-1} Z_{X_{i,m,j}} \left(q^{\frac{m}{d}} T\right) \right) \right)^{(-1)^{n+1}}.$$

Let $W = W_1^{\alpha_1} \cdot W_2^{\alpha_2} \cdot \ldots \cdot W_s^{\alpha_s}$ be the decomposition of W into product of irreducible polynomials $W_i \in \mathbb{Z}[q^{\frac{1}{d}}, T]$, $\alpha_i \in \mathbb{Z}$. Then

$$Z_q\left(\widetilde{\mathcal{X}_{d,n}}\right)(T) = \prod \left\{ W_i^{\alpha_1} : W_i \in \mathbb{Z}[q,T],\ i = 1, \dots, s \right\}.$$

$\square$

Therefore, in order to compute zeta function of $\widetilde{\mathcal{X}_{d,n}}$ for any $i \in \{1, 2, \dots, n\}$ it is enough to produce the following tables

$k \backslash j$	0	1	$\cdots$	j	$\cdots$	$d-1$
0	$Z_{X_i,0,0}$	$Z_{X_i,0,1}$		$Z_{X_i,0,j}$		$Z_{X_i,0,d-1}$
1	$Z_{X_i,1,0}$	$Z_{X_i,1,1}$		$Z_{X_i,1,j}$		$Z_{X_i,1,d-1}$
2	$Z_{X_i,2,0}$	$Z_{X_i,2,1}$	$\cdots$	$Z_{X_i,2,j}$	$\cdots$	$Z_{X_i,2,d-1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$d-1$	$Z_{X_i,d-1,0}$	$Z_{X_i,d-1,1}$		$Z_{X_i,d-1,j}$		$Z_{X_i,d-1,d-1}$
				$\Vert$ $v_{X_i,j}$		

Table 2.2: $Z_{X_i,j,k}$

for any considered variety X_i. Then we we evaluate vector $v_{X_i,j}$ on

$$v_d := \left(T, \sqrt[d]{q}T, \sqrt[d]{q^2}T, \dots, \sqrt[d]{q^{d-1}}T\right)$$

and multiply all its terms. Then we take tensor product for all $i \in \{1, 2, \dots, n\}$ and take product over $j \in \{0, 1, \dots, d-1\}$. Finally we take $(-1)^{n+1}$ power of the result and form product of factors containing only integral powers of q.

§ 2.8 Rational elliptic surfaces

In this section we collect basic information about rational elliptic surfaces. We refer to [Mir89], [IS96], [SS10], [SS19] for a complete exposition in this topic.

Definition 2.8.1. An *elliptic surface* is a smooth surface S with a proper morphism $f : S \to B$ onto a smooth curve B such that generic fiber has genus one.

An elliptic surface $f : S \to B$ is *relatively minimal* if there are no (-1)-curves contained in fibers of f. By blowing down (-1)-curves in fibers of S we obtain relatively minimal surface birational to S.

If S is an elliptic surface then $K_S^2 = 0$ and consequently S is *rational* (i.e. birational to $\mathbb{P}^2$) iff $B \simeq \mathbb{P}^1$ and $e(S) = 12$ (equivalently the sum of Euler characteristic of fibers equals 12).

Every rational elliptic surface has a singular fiber. The complete list of possible types of singular fibers was given by Kodaira in [Kod63]:

- irreducible $\mathbf{I}_1$ and **II**,
- reducible infinite series $\mathbf{I}_n$ $(n > 0)$, $\mathbf{I}_n^*$ $(n \geq 0)$,
- reducible five types **III**, **IV**, $\mathbf{II}^*$, $\mathbf{III}^*$ and $\mathbf{IV}^*$.
- multiple fibers.

We shall consider only elliptic surfaces with a fixed section, as the intersection of a section with any fiber equals one, an elliptic surfaces with a section has no multiple fibers.

Fibers $\mathbf{I}_n$ are called *semistable*.

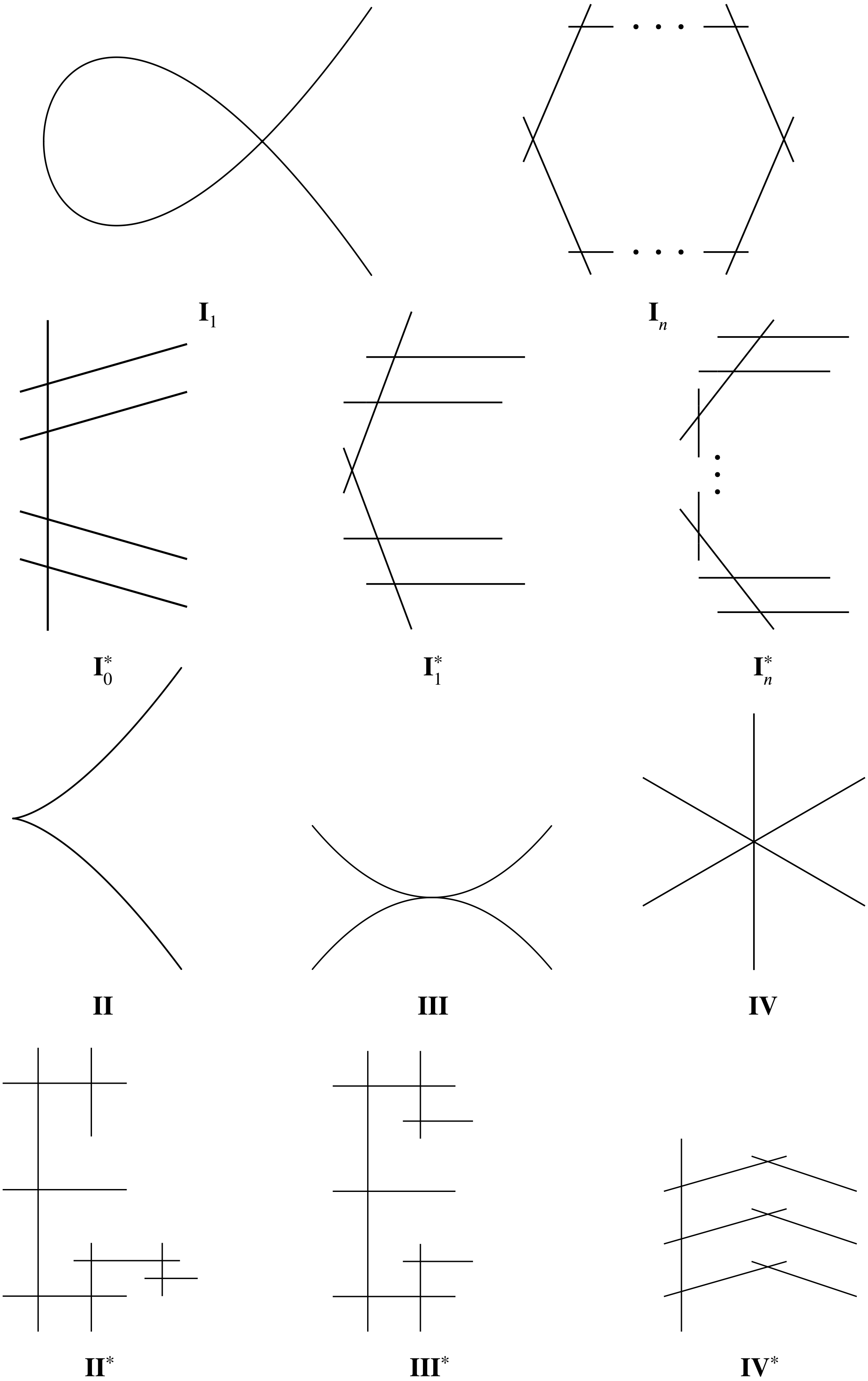


Figure 2.1: Kodaira classification of possible singular fibers

S. Herfurtner in [Her91] gave a complete list of rational elliptic surfaces with four singular fibers and non-constant J-invariant. It was obtained by analysing the possible combinations of singular fibers and then listing all Weierstrass models of such elliptic fibrations.

In [MP86] similar classification was given for *extremal rational elliptic surfaces* i.e. rational elliptic surface with finite Mordell-Weil group and Picard number equal to $h^{1,1}$. Six of them (in fact all possible with semistable fibers) $(\mathbf{I}_9, \mathbf{I}_1, \mathbf{I}_1, \mathbf{I}_1)$, $(\mathbf{I}_8, \mathbf{I}_2, \mathbf{I}_1, \mathbf{I}_1)$, $(\mathbf{I}_6, \mathbf{I}_3, \mathbf{I}_2, \mathbf{I}_1)$, $(\mathbf{I}_5, \mathbf{I}_5, \mathbf{I}_1, \mathbf{I}_1)$, $(\mathbf{I}_4, \mathbf{I}_4, \mathbf{I}_2, \mathbf{I}_2)$, $(\mathbf{I}_3, \mathbf{I}_3, \mathbf{I}_3, \mathbf{I}_3)$ were studied by Beauville in [Bea82] – these surfaces are called *Beauville surfaces*.

Chapter 3

Cynk-Hulek varieties

In this chapter we briefly recall the original Cynk-Hulek construction of an arbitrary dimensional Calabi-Yau manifolds involving elliptic curves with automorphism of order 2, 3 and 4. Then, we shall prove that this construction can be also carried out for elliptic curves with automorphism of order 6. In the proof we shall use toric methods introduced in chapter 2.

In the last section we shall compute Hodge numbers and the Zeta function of all considered Calabi-Yau manifolds.

§ 3.1 Construction

Let E_d be an elliptic curve with an order d automorphism $\phi_d : E_d \to E_d$, then $d = 2, 3, 4, 6$ (see [Sil09]). Up to an isomorphism the curve E_d and the automorphism ϕ_d can be given as:

- E_2 is an arbitrary elliptic curve, and the involution ϕ_2 is the $[-1]$ map

$$\phi_2(x, y) = (x, -y),$$

- E_3 has the Weierstrass equation $y^2 = x^3 + 1$, and the automorphism ϕ_3 is given by

$$\phi_3(x, y) = (\zeta_3 x, y),$$

  where ζ_3 denotes a fixed 3-rd root of unity,

- E_4 has the Weierstrass equation $y^2 = x^3 + x$, and the automorphism ϕ_4 is given by

$$\phi_4(x, y) = (-x, iy),$$

- E_6 has the Weierstrass equation $y^2 = x^3 + 1$, and the automorphism ϕ_6 is given by

$$\phi_6(x, y) = (\zeta_6^2 x, -y),$$

where ζ_6 denotes a fixed 6-th root of unity satisfying $\zeta_6^2 = \zeta_3$.

For $d \in \{2, 3, 4, 6\}$ and any positive integer n, the following group

$$G_{d,n} := \{(m_1, m_2, \ldots, m_n) \in \mathbb{Z}_d^n : m_1 + m_2 + \ldots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

acts on E_d^n by $\phi_d^{m_i}$ on the i-th factor.

Note that $G_{d,n}$ preserves the canonical bundle $\omega_{E_d^n}$ of the manifold E_d^n. Therefore it is natural to ask whether variety ${E_d^n}/{G_{d,n}}$ admits crepant resolution.

Theorem 3.1.1 ([CH07])**.** *If $d = 2, 3, 4$, then there exists a crepant resolution*

$$\widetilde{{E_d^n}/{G_{d,n}}} \to {E_d^n}/{G_{d,n}}.$$

Consequently, $X_{d,n} := \widetilde{{E_d^n}/{G_{d,n}}}$ is an n-dimensional Calabi-Yau manifold.

The above construction for $n = 3$ and $d = 2$ i.e. ${E_2^3}/{\mathbb{Z}_2^2}$ was first considered by C. Borcea in [Bor97]. He also proved that $\widetilde{{E_2^3}/{\mathbb{Z}_2^2}}$ has CM iff E_2 has CM. Moreover A. Molnar in his PhD ([Mol15]) focused on modular aspects of quotients of E_d^3 by different finite groups.

We shall proof 3.1.1 using toric techniques introduced in Chapter 2.

3.1.1 $d = 2$

Let X_1, X_2 be two Calabi-Yau manifolds with involutions $\eta_i : X_i \to X_i$ (for $i = 1, 2$) such that

$$\eta_1^*\left(\omega_{X_1}\right) = -\omega_{X_1} \quad \text{and} \quad \eta_2^*\left(\omega_{X_2}\right) = -\omega_{X_2},$$

where ω_{X_i} denotes a chosen generator of $H^{n,0}(X_i)$, for $i = 1, 2$.

Suppose that the fixed point loci $\mathrm{Fix}(\eta_1)$ and $\mathrm{Fix}(\eta_2)$ of η_1 and η_2 respectively, are disjoint union of smooth divisors, in particular both have linearisation of the form $(-1, 1, 1, \ldots, 1)$ near any point of $\mathrm{Fix}(\eta_1)$ or $\mathrm{Fix}(\eta_2)$.

Proposition 3.1.2. *Under the above assumptions the quotient variety ${X_1 \times X_2}/{\mathbb{Z}_2}$ has a resolution of singularities $\widetilde{{X_1 \times X_2}/{\mathbb{Z}_2}}$, which is a Calabi-Yau manifold. Moreover, the manifold $\widetilde{{X_1 \times X_2}/{\mathbb{Z}_2}}$, admits a $\mathbb{Z}_2$-action which satisfies the same assumptions as for X_2.*

Proof. The automorphism $\eta := \eta_1 \times \eta_2$ has a local linearisation around fixed point of one of the following types: $\frac{1}{2}(1,1)$.

In local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(x^2, \frac{y}{x}\right), \left(\frac{x}{y}, y^2\right),$$

hence $\mathrm{id} \times \eta_2$ lifts to resolution as $(1,-1)$ and $(-1,1)$, respectively. Consequently the action on $\widetilde{X_1 \times X_2 /_{\eta}}$ induced by $\mathrm{id} \times \eta_2$ satisfies the assumptions we made on the action η_2, hence the statement follows. $\square$

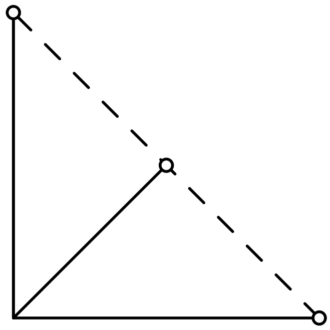


Figure 3.1: Decomposition of $\frac{1}{2}(1,1)$

The case where X_1 is a $K3$ surface and X_2 elliptic curve, both of them admitting non-symplectic involution was studied independently by Borcea ([Bor97]) and Voisin ([Voi93]).

3.1.2 $d = 3$

Let X_1, X_2 be two Calabi-Yau manifolds with automorphisms $\eta_i : X_i \to X_i$ (for $i = 1,2$) of order 3 such that

$$\eta_1^* \left(\omega_{X_1}\right) = \zeta_3 \omega_{X_1} \quad \text{and} \quad \eta_2^* \left(\omega_{X_2}\right) = \zeta_3^2 \omega_{X_2},$$

where ω_{X_i} denotes a chosen generator of $H^{n,0}(X_i)$, for $i = 1,2$. Suppose that

1. the fixed point locus $\mathrm{Fix}(\eta_1)$ of η_1 is a disjoint union of smooth divisors, in particular η_1 has linearisation of the form $(\zeta_3, 1, 1, \ldots, 1)$ near any point of $\mathrm{Fix}(\eta_1)$,

2. $\mathrm{Fix}(\eta_2)$ is a disjoint union of submanifolds of codimension at most 2. In particular η_2 has linearisation of the form

   - $(\zeta_3^2, 1, 1, \ldots, 1)$ near a component of codimension one of $\mathrm{Fix}(\eta_2)$,

- $(\zeta_3, \zeta_3, 1, 1, \dots, 1)$ near a component of codimension two of $\mathrm{Fix}(\eta_2)$.

Proposition 3.1.3. *Under the above assumptions the quotient variety* $X_1 \times X_2 / \mathbb{Z}_3$ *has a resolution of singularities* $\widetilde{X_1 \times X_2 / \mathbb{Z}_3}$*, which is a Calabi-Yau manifold. Moreover, the manifold* $\widetilde{X_1 \times X_2 / \mathbb{Z}_3}$*, admits a* $\mathbb{Z}_3$*-action which satisfies the same assumptions as for* X_2*.*

Proof. The automorphism $\eta := \eta_1 \times \eta_2$ has a local linearisation around fixed point of one of the following types: $\frac{1}{3}(1,2)$ or $\frac{1}{3}(1,1,1)$. Both singularities can be resolved by subdividing junior simplexes in dimension 2 and 3 (figure: 3.2).

(i) If η has a local linearisation given by $\frac{1}{3}(1,2)$ near $\mathrm{Fix}(\eta)$, then in local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(x^3, \frac{y}{x^2}\right), \left(\frac{x^2}{y}, \frac{y^2}{x}\right), \left(\frac{x}{y^2}, y^3\right).$$

The action of $\mathrm{id} \times \eta_2$ has a linearisation $(1, \zeta_3^2, 1, \dots, 1)$, so it lifts to the resolution as $(1, \zeta_3^2)$, (ζ_3, ζ_3), $(\zeta_3^2, 1)$, respectively.

(ii) If η has a local linearisation given by $\frac{1}{3}(1,1,1)$ near $\mathrm{Fix}(\eta)$, then in local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(x^3, \frac{y}{x}, \frac{z}{x}\right), \left(\frac{x}{y}, y^3, \frac{z}{y}\right), \left(\frac{x}{z}, \frac{y}{z}, z^3\right),$$

consequently $\mathrm{id} \times \eta_2$ lifts to the resolution as: $(1, \zeta_3, \zeta_3)$, $(\zeta_3^2, 1, 1)$, $(\zeta_3^2, 1, 1)$, respectively.

In all considered cases the action on $\widetilde{X_1 \times X_2 / \eta}$ induced by $\mathrm{id} \times \eta_2$ satisfies the assumptions we made on the action η_2, hence the statement follows.

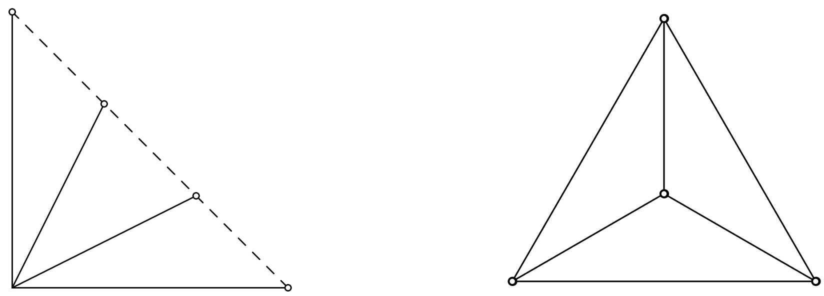


Figure 3.2: Decomposition for $\frac{1}{3}(1,2)$ and $\frac{1}{3}(1,1,1)$

□

3.1.3 $d = 4$

Let X_1, X_2 be two Calabi-Yau manifolds with automorphisms $\eta_i : X_i \to X_i$ (for $i = 1, 2$) of order 4 such that

$$\eta_1^*\left(\omega_{X_1}\right) = \zeta_4\omega_{X_1} \quad \text{and} \quad \eta_2^*\left(\omega_{X_2}\right) = \zeta_4^3\omega_{X_2},$$

where ω_{X_i} denotes a chosen generator of $H^{n,0}(X_i)$, for $i = 1, 2$. Suppose that

1. the fixed point locus $\mathrm{Fix}(\eta_1)$ of η_1 is a disjoint union of smooth divisors, in particular η_1 has linearisation of the form $(\zeta_4, 1, 1, \dots, 1)$ near any point of $\mathrm{Fix}(\eta_1)$,
2. $\mathrm{Fix}(\eta_2)$ is a disjoint union of submanifolds of codimension at most 3. In particular η_2 has linearisation of the form
   - $(\zeta_4^3, 1, 1, \dots, 1)$ near a component of codimension one of $\mathrm{Fix}(\eta_2)$,
   - $(\zeta_4, \zeta_4^2, 1, 1, \dots, 1)$ near a component of codimension two of $\mathrm{Fix}(\eta_2)$,
   - $(\zeta_4, \zeta_4, \zeta_4, 1, 1, \dots, 1)$ near a component of codimension three of $\mathrm{Fix}(\eta_2)$.

Proposition 3.1.4. *Under the above assumptions the quotient variety* $X_1 \times X_2 / \mathbb{Z}_4$ *has a resolution of singularities* $\widetilde{X_1 \times X_2 / \mathbb{Z}_4}$, *which is a Calabi-Yau manifold. Moreover, the manifold* $\widetilde{X_1 \times X_2 / \mathbb{Z}_4}$, *admits a* $\mathbb{Z}_4$*-action which satisfies the same assumptions as for* X_2.

Proof. The automorphism $\eta := \eta_1 \times \eta_2$ has a local linearisation around fixed point of one of the following types: $\frac{1}{4}(1,3)$, $\frac{1}{4}(1,1,2)$ or $\frac{1}{4}(1,1,1,1)$. Resulting singularities can be resolved by subdividing junior simplexes in dimension 2, 3 and 4 (figure 3.3).

(i) If η has a local linearisation given by $\frac{1}{4}(1,3)$ near $\mathrm{Fix}(\eta)$, then in local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(x^4, \frac{y}{x^3}\right), \left(\frac{x^3}{y}, \frac{y^2}{x^2}\right), \left(\frac{x^2}{y^2}, \frac{y^3}{x}\right), \left(\frac{x}{y^3}, y^4\right).$$

The action of $\mathrm{id} \times \eta_2$ has a linearisation $(1, \zeta_4^3, 1, \dots, 1)$, so it lifts to the resolution as $(1, \zeta_4^3)$, (ζ_4, ζ_4^2), (ζ_4^2, ζ_4), $(\zeta_4^3, 1)$,respectively.

(ii) If η has a local linearisation given by $\frac{1}{4}(1,1,2)$ near $\mathrm{Fix}(\eta)$, then in local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(\frac{z}{x^2}, \frac{y}{x}, x^4\right), \left(z^2, \frac{y}{x}, \frac{x^2}{z}\right), \left(z^2, \frac{y^2}{z}, \frac{x}{y}\right), \left(y^4, \frac{x}{y}, \frac{z}{y^2}\right),$$

consequently $\mathrm{id} \times \eta_2$ lifts to the resolution as: $(\zeta_4^2, \zeta_4, 1)$, $(1, \zeta_4, \zeta_4^2)$, $(1, 1, \zeta_4^3)$, $(1, \zeta_4^3, 1)$, respectively.

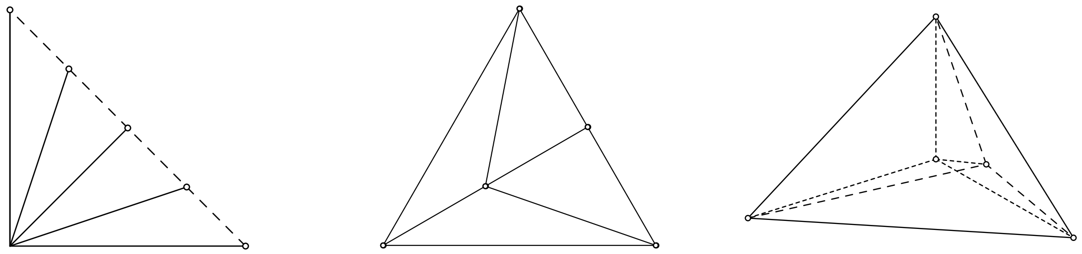


Figure 3.3: Decompositions for $\frac{1}{4}(1,3)$, $\frac{1}{4}(1,1,2)$ and $\frac{1}{4}(1,1,1,1)$

In all considered cases the action on $\widetilde{X_1 \times X_2/_{\eta}}$ induced by $\mathrm{id} \times \eta_2$ satisfies the assumptions we made on the action η_2, hence the statement follows. □

§ 3.2 $d = 6$

In [CH07] a crepant resolution of $X_{n,d}$ for $d = 2, 3, 4$ was constructed by an iterated approach. In this section we will give the missing construction for elliptic curves admitting automorphisms of order 6 and prove that there exists a crepant resolution of these manifolds.

3.2.1 Resolution of singularities

Let X_1, X_2 be two Calabi-Yau manifolds with automorphisms $\eta_i : X_i \to X_i$ (for $i = 1, 2$) of order 6 such that

$$\eta_1^* \left(\omega_{X_1}\right) = \zeta_6 \omega_{X_1} \quad \text{and} \quad \eta_2^* \left(\omega_{X_2}\right) = \zeta_6^5 \omega_{X_2},$$

where ω_{X_i} denotes a chosen generator of $H^{n,0}(X_i)$, for $i = 1, 2$.

Assume that:

1. the fixed point locus $\mathrm{Fix}(\eta_1)$ of η_1 is a disjoint union of smooth divisors, in particular η_1 has linearisation of the form $(\zeta_6, 1, 1, \ldots, 1)$ near any point of $\mathrm{Fix}(\eta_1)$,

2. $\mathrm{Fix}(\eta_2)$ is a disjoint union of submanifolds of codimension at most 3. In particular η_2 has linearisation of the form

    - $(\zeta_6^5, 1, 1, \ldots, 1)$ near a component of codimension one of $\mathrm{Fix}(\eta_2)$,

- $(\zeta_6^4, \zeta_6, 1, 1, \dots, 1)$ or $(\zeta_6^3, \zeta_6^2, 1, 1, \dots, 1)$ near a component of codimension two of $\mathrm{Fix}(\eta_2)$,

3. $\mathrm{Fix}(\eta_1^2) \setminus \mathrm{Fix}(\eta_1)$ is a disjoint union of smooth divisors in particular η_1^2 has linearisation $(\zeta_3, 1, 1, \dots, 1)$ along any component of $\mathrm{Fix}(\eta_1^2) \setminus \mathrm{Fix}(\eta_1)$,

4. $\mathrm{Fix}(\eta_1^3) \setminus \mathrm{Fix}(\eta_1)$ is a disjoint union of smooth divisors in particular η_1^3 has linearisation $(-1, 1, 1, \dots, 1)$ along any component of $\mathrm{Fix}(\eta_1^3) \setminus \mathrm{Fix}(\eta_1)$,

5. $\mathrm{Fix}(\eta_2^2) \setminus \mathrm{Fix}(\eta_2)$ is a disjoint union of smooth submanifolds of codimension at most 2, so η_2^2 has linearisation of the form $(\zeta_3^2, 1, 1, \dots, 1)$ or $(\zeta_3, \zeta_3, 1, 1, \dots, 1)$ along any component of $\mathrm{Fix}(\eta_2^2) \setminus \mathrm{Fix}(\eta_2)$,

6. $\mathrm{Fix}(\eta_2^3) \setminus \mathrm{Fix}(\eta_2)$ is a disjoint union of smooth divisors, so η_2^3 has linearisation of the form $(-1, 1, 1, \dots, 1)$ along any component of $\mathrm{Fix}(\eta_2^3) \setminus \mathrm{Fix}(\eta_2)$,

7. the automorphism η_2 has a local linearisation of the form $(\zeta_6^2, \zeta_6^2, \zeta_6, 1, 1, \dots, 1)$ along any codimensional 3 component of $\mathrm{Fix}(\eta_2)$.

We have the following:

Proposition 3.2.1 ([Bur20]). *Under the above assumptions the quotient* $X_1 \times X_2 / \eta_1 \times \eta_2$ *of the product* $X_1 \times X_2$ *by the action of* $\eta_1 \times \eta_2$ *admits a crepant resolution of singularities* $\widetilde{X_1 \times X_2 / \eta_1 \times \eta_2}$*. Furthermore* $\mathrm{id} \times \eta_2$ *induces an automorphism of order 6 on* $\widetilde{X_1 \times X_2 / \eta_1 \times \eta_2}$ *that satisfies all assumption we put on* η_2.

Proof. By the assumption we made, the automorphism $\eta := \eta_1 \times \eta_2$ has a local linearisation around any fixed point of one of the following types:

(i) $(\zeta_6, \zeta_6^5, 1, 1, \dots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1, 5)$,

(ii) $(\zeta_6, \zeta_6, \zeta_6^4, 1, 1, \dots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1, 1, 4)$,

(iii) $(\zeta_6, \zeta_6^2, \zeta_6^3, 1, 1, \dots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1, 2, 3)$,

(iv) $(\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \dots, 1)$ which corresponds to singularity of type $\frac{1}{6}(1, 1, 2, 2)$.

We shall use suitable resolution of the cyclic singularity in each case.

(i) If η has a local linearisation given by $(\zeta_6, \zeta_6^5, 1, 1, \dots, 1)$ near $\mathrm{Fix}(\eta)$, then in local coordinates, the map from $X_1 \times X_2$ to the resolution is given in affine charts by

$$\left(x^6, \frac{y}{x^5}\right), \left(\frac{x^5}{y}, \frac{y^2}{x^4}\right), \left(\frac{x^4}{y^2}, \frac{y^3}{x^3}\right), \left(\frac{x^3}{y^3}, \frac{y^4}{x^2}\right), \left(\frac{x^2}{y^4}, \frac{y^5}{x}\right) \text{ or } \left(\frac{x}{y^5}, y^6\right).$$

The action of $\mathrm{id} \times \eta_2$ has a linearisation $(1, \zeta_6^5, 1, \dots, 1)$, so it lifts to the resolution as $(1, \zeta_6^5)$, (ζ_6, ζ_6^4), (ζ_6^2, ζ_6^3), (ζ_6^3, ζ_6^2), (ζ_6^4, ζ_6) and $(\zeta_6^5, 1)$, respectively.

(ii) If η has a local linearisation given by $(\zeta_6, \zeta_6, \zeta_6^4, 1, 1, \dots, 1)$ near $\mathrm{Fix}(\eta)$, then we can use a toric resolution of $\frac{1}{6}(1, 1, 4)$ singularity described in 2.1.23. Thus the map from $X_1 \times X_2$ to the resolution is given in affine charts as

$$\left(x^6, \frac{z}{x^4}, \frac{y}{x}\right), \quad \left(\frac{x^4}{z}, \frac{z^2}{x^2}, \frac{y}{x}\right), \quad \left(\frac{x^2}{z^2}, z^3, \frac{y}{x}\right), \quad \left(\frac{x}{y}, \frac{z}{y^4}, y^6\right),$$
$$\left(\frac{x}{y}, \frac{z^2}{y^2}, \frac{y^4}{z}\right) \quad \text{or} \quad \left(\frac{x}{y}, z^3, \frac{y^2}{z^2}\right).$$

Therefore the action of $\mathrm{id} \times \eta_2$ lifts to the resolution as $(1, \zeta_6^4, \zeta_6)$, $(\zeta_6^2, \zeta_6^2, \zeta_6)$, $(\zeta_6^4, 1, \zeta_6)$, $(\zeta_6^5, 1, 1)$, $(\zeta_6^5, 1, 1)$, $(\zeta_6^5, 1, 1)$.

(iii) If η has a local linearisation given by $(\zeta_6, \zeta_6^2, \zeta_6^3, 1, 1, \dots, 1)$ near $\mathrm{Fix}(\eta)$, then we use again toric resolution of $\frac{1}{6}(1, 2, 3)$ singularity. There are five different decompositions of junior simplex which give a toric resolution i.e:

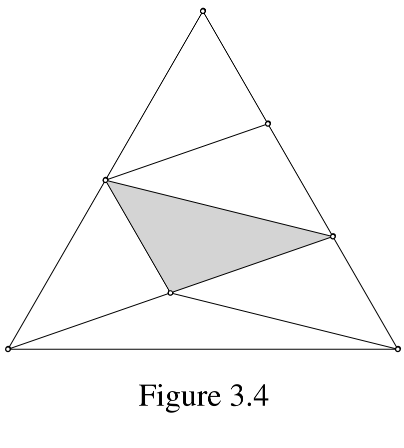


Figure 3.4

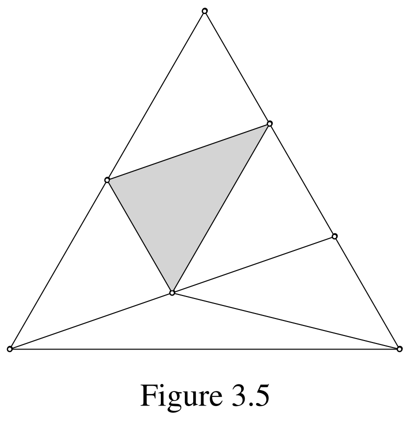


Figure 3.5

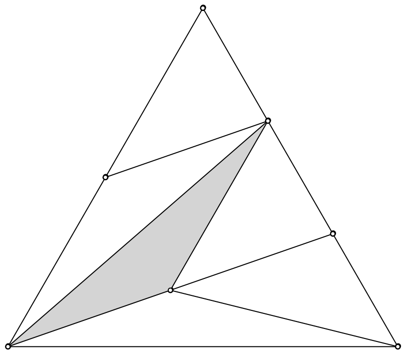


Figure 3.6

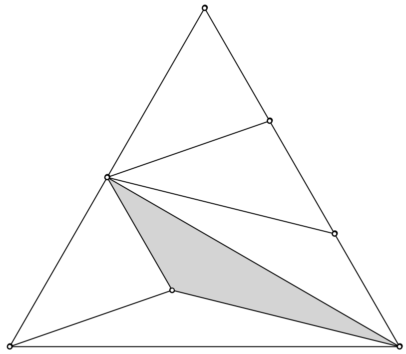


Figure 3.7

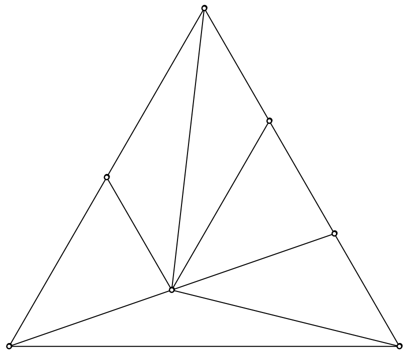


Figure 3.8

Figure 3.9: Possible triangulations of the junior simplex of $\frac{1}{6}(1,2,3)$

Note that in 3.4 the local affine chart corresponds to marked grey triangle is given by

$$\left(\frac{yz^2}{x^2}, \frac{xy}{z}, \frac{x^2}{y}\right),$$

so the action of $\mathrm{id} \times \eta_2$ lifts to that affine chart as $(\zeta_6^2, \zeta_6^5, \zeta_6^4)$, which is excluded by the condition 7.

Similarly, local affine charts corresponding to marked grey triangles in figures 3.5, 3.6, 3.7 are equal to respectively

$$\left(\frac{y^2z}{x}, \frac{xy}{z}, \frac{xz}{y^2}\right), \quad \left(\frac{xz}{y^2}, \frac{y^4}{x^2}, \frac{x^2}{y}\right), \quad \left(\frac{xy}{z}, \frac{z^3}{x^3}, \frac{x^3}{z}\right),$$

with liftings $(\zeta_6, \zeta_6^5, \zeta_6^5)$, $(\zeta_6^5, \zeta_6^2, \zeta_6^4)$, $(\zeta_6^5, \zeta_6^3, \zeta_6^3)$, respectively – all of them are excluded by 7.

Only 3.8 is suitable for our considerations. For that chosen triangulation (and hence resolution), the map from $X_1 \times X_2$ to the resolution is given in affine charts as (see

3.10):

$$\left(x^6, \frac{z}{x^3}, \frac{y}{x^2}\right), \quad \left(\frac{x^3}{z}, z^2, \frac{y}{x^2}\right), \quad \left(\frac{x^2}{y}, z^2, \frac{y^2}{xz}\right), \quad \left(\frac{xz}{y^2}, z^2, \frac{y^3}{z^2}\right),$$
$$\left(\frac{z^2}{y^3}, y^3, \frac{xy}{z}\right) \quad \text{or} \quad \left(\frac{z}{xy}, y^3, \frac{x^2}{y}\right).$$

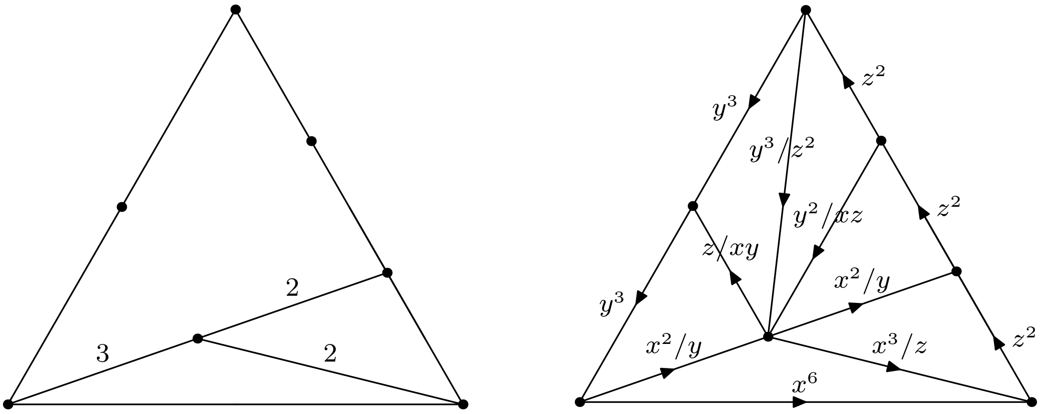


Figure 3.10: Crepant resolution of $\frac{1}{6}(1,2,3)$

The action of $\mathrm{id} \times \eta_2$ has a local linearisation $(1, \zeta_6^2, \zeta_6^3, 1, \ldots, 1)$, hence it lifts to the resolution as $(1, \zeta_6^3, \zeta_6^2)$, $(\zeta_6^3, 1, \zeta_6^2)$, $(\zeta_6^4, 1, \zeta_6)$, $(\zeta_6^5, 1, 1)$, $(1, 1, \zeta_6^5)$, $(\zeta_6, 1, \zeta_6^4)$, respectively.

(iv) If η has a local linearisation given by $(\zeta_6, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$ near $\mathrm{Fix}(\eta_2)$, then the map is given by

$$\left(x^6, \frac{y}{x}, \frac{z}{x^2}, \frac{t}{x^2}\right), \quad \left(\frac{x^2}{z}, \frac{y}{x}, z^3, \frac{t}{z}\right), \quad \left(\frac{x^2}{t}, \frac{y}{x}, t^3, \frac{z}{t}\right), \quad \left(\frac{x}{y}, y^6, \frac{z}{y^2}, \frac{t}{y^2}\right),$$
$$\left(\frac{x}{y}, \frac{y^2}{z}, z^3, \frac{t}{z}\right) \quad \text{or} \quad \left(\frac{x}{y}, \frac{y^2}{t}, \frac{z}{t}, t^3\right).$$

The action of $\mathrm{id} \times \eta_2$ has a local linearisation $(1, \zeta_6, \zeta_6^2, \zeta_6^2, 1, 1, \ldots, 1)$, hence it lifts to the resolution as $(1, \zeta_6, \zeta_6^2, \zeta_6^2)$, $(\zeta_6^4, \zeta_6, 1, 1)$, $(\zeta_6^4, \zeta_6, 1, 1)$, $(\zeta_6^5, 1, 1, 1)$, $(\zeta_6^5, 1, 1, 1)$, $(\zeta_6^5, 1, 1, 1)$.

In all considered cases the action on $\widetilde{X_1 \times X_2/_{\eta}}$ induced by $\mathrm{id} \times \eta_2$ satisfies the assumptions we made on the action η_2.

Finally near the points of $\mathrm{Fix}(\eta^2) \setminus \mathrm{Fix}(\eta)$ and $\mathrm{Fix}(\eta^3) \setminus \mathrm{Fix}(\eta)$ we first consider the quotient $X_1 \times X_2/_{\eta^2}$ $\left(\text{resp. } X_1 \times X_2/_{\eta^3}\right)$, then using Prop. 2.1 and 3.1 of [CH07] we construct

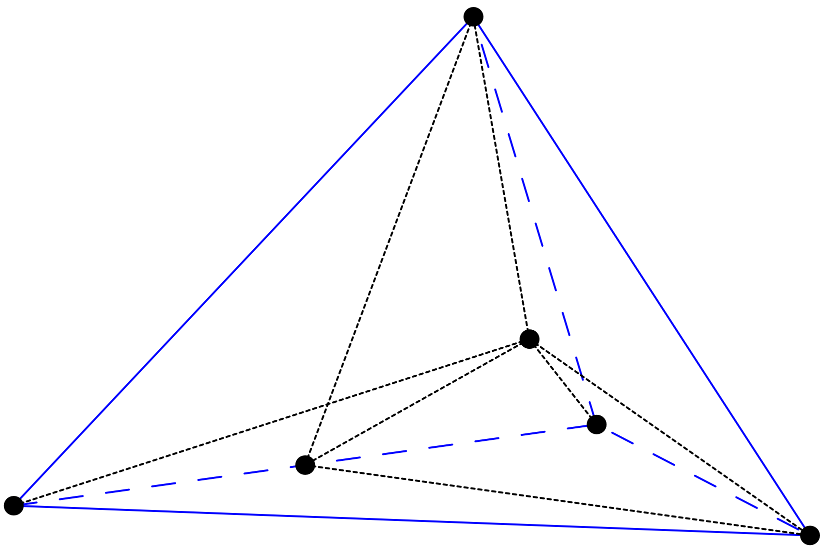


Figure 3.11: Junior symplex of $\frac{1}{6}(1,1,2,2)$

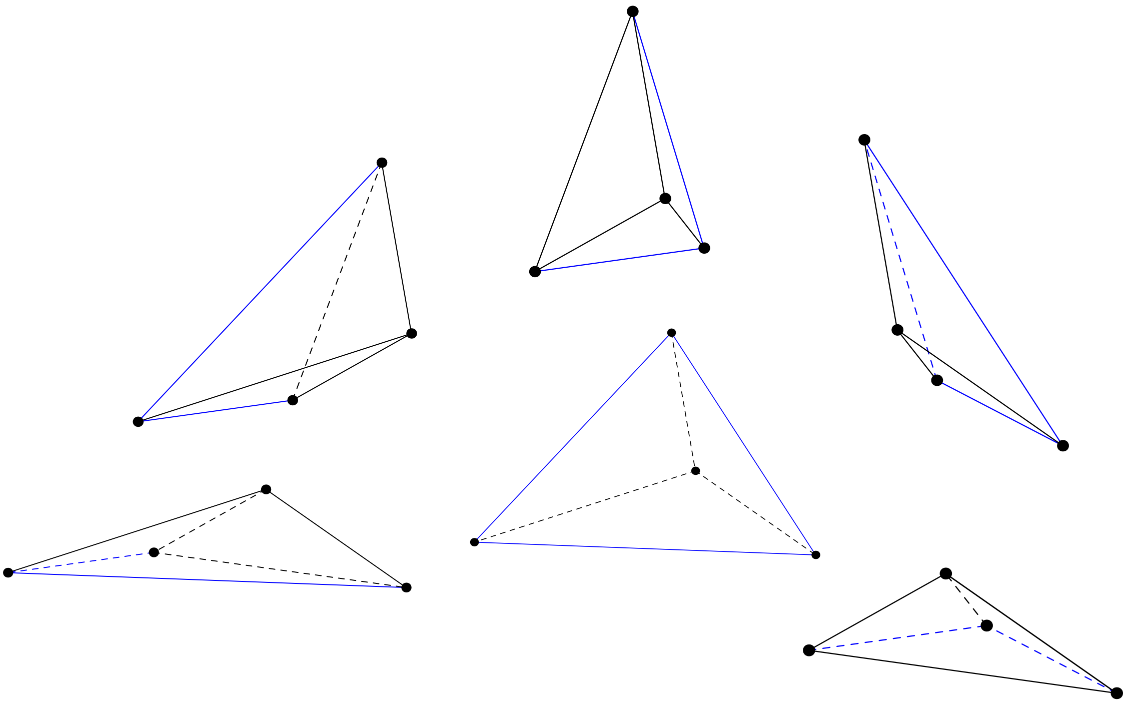


Figure 3.12: Triangulation of 3.11

crepant resolutions of

$$\widetilde{\left(X_1 \times X_2 \big/ \eta^2\right)} \big/ \eta^3 \qquad \text{resp. } \widetilde{\left(X_1 \times X_2 \big/ \eta^3\right)} \big/ \eta^2.$$

$\square$

We can iterate the procedure in Proposition 3.2.1. Consider Calabi-Yau manifolds X_1, $X_2, \dots, X_n$ with automorphisms ϕ_i of order 6 such that

- $\phi_i^*(\omega_{X_i}) = \zeta_6 \omega_{X_i}$ where ω_{X_i} is a canonical form on X_i,
- ϕ_1 satisfies the assumptions we put on η_1 in 3.2.1,
- ϕ_i^5 satisfies, for $i = 2, \dots, n$, the assumptions we put on η_2 in 3.2.1.

The group $G_{6,n}$ acts on $X_1 \times X_2 \times \dots \times X_n$ as

$$\left(\phi_1^{m_1}(x_1), \phi_2^{m_2}(x_2), \dots, \phi_n^{m_n}(x_n)\right)$$

for $(m_1, m_2, \dots, m_n) \in G_{6,n}$ and $x_i \in X_i$ for $i = 1, 2, \dots, n$.

Proposition 3.2.2. *The quotient of the product $X_1 \times X_2 \times \dots \times X_n$ by the action of $G_{6,n}$ has a crepant resolution of singularities which is a Calabi-Yau manifold and such that the action of $\mathbb{Z}_6^n$ on $X_1 \times X_2 \times \dots \times X_n$ lifts to a purely non-symplectic action of $\mathbb{Z}_6$ on this resolution.*

Proof. For $n = 2$ this is Proposition 3.2.1. For an inductive approach notice that

$$X_1 \times X_2 \times \dots \times X_n \big/ G_{6,n} \simeq \left(X_1 \times X_2 \times \dots \times X_{n-1} \big/ G_{6,n-1}\right) \times X_n \big/ \mathbb{Z}_6.$$

By the inductive hypothesis the quotient $X_1 \times X_2 \times \dots \times X_{n-1} \big/ G_{6,n-1}$ has a crepant resolution $\widetilde{X}$ and the action of $G_{6,n-1}$ lifts to $\widetilde{X}$ as a purely non-symplectic action of $\mathbb{Z}_6$. Using Proposition 3.2.1 again we conclude the proof. □

As a special case we get:

Theorem 3.2.3. *There exists a crepant resolution*

$$\widetilde{E_6^n \big/ G_{6,n}} \to E_6^n \big/ G_{6,n}.$$

Consequently, $X_{6,n} := \widetilde{E_6^n \big/ G_{6,n}}$ is an n-dimensional Calabi-Yau manifold.

Remark 3.2.4. In the constructed crepant resolution of $E_6^n \big/ G_{6,n}$ we need a suitable toric resolution, as the iterated approach in [CH07] leads to a local action of type $(\zeta_6, \zeta_6, \zeta_6^5, \zeta_6^5)$, which has no junior elements and so the quotients does not admit any crepant resolution.

Also, we were not able to use a factorisation of an action of order 6 into an action of order 2 and 3. Indeed the second power of the action (iv) in the proof of Proposition 3.2.1 is equal to $(\zeta_3, \zeta_3, \zeta_3^2, \zeta_3^2, 1, \dots)$, which again has no junior element and consequently no crepant resolution. The third power of the action (iii) equals $(-1, 1, -1, 1, \dots)$ and the factorization into an action of order 2 followed by an action of order 3 gives inverse local chart $\left(\frac{x^3}{z}, \frac{xy}{z}, \frac{z^3}{x^3}\right)$. In this chart the action of η_1 lifts to the resolution as $(-1, \zeta_6, -1)$. In the next step we get an action $(\zeta_6, \zeta_6^3, \zeta_6^5, \zeta_6^5)$ with the third power equal to $(-1, -1, -1, -1)$, which clearly has no crepant resolution.

§ 3.3 Hodge numbers

In order to compute Hodge diamonds of constructed varities we will use methods described in 2.3.

3.3.1 $d = 6$

From table 5.3 and theorem 2.3.3 we get:

$$
\begin{aligned}
&h^{p,q}\left(\widetilde{E_6^n/G_{6,n}}\right) = \\
&= \Bigg\{ \left((1+XY)\cdot\sqrt[6]{(XY)^0} + 1\cdot\sqrt[6]{XY} + 2\cdot\sqrt[6]{(XY)^2} + 2\cdot\sqrt[6]{(XY)^3} + 2\cdot\sqrt[6]{(XY)^4} + 1\cdot\sqrt[6]{(XY)^5}\right)^n + \\
&+ \left(X\cdot\sqrt[6]{(XY)^0} + 0\cdot\sqrt[6]{XY} + 0\cdot\sqrt[6]{(XY)^2} + 0\cdot\sqrt[6]{(XY)^3} + 0\cdot\sqrt[6]{(XY)^4} + 0\cdot\sqrt[6]{(XY)^5}\right)^n + \\
&+ \left(0\cdot\sqrt[6]{(XY)^0} + 0\cdot\sqrt[6]{XY} + 0\cdot\sqrt[6]{(XY)^2} + 1\cdot\sqrt[6]{(XY)^3} + 0\cdot\sqrt[6]{(XY)^4} + 0\cdot\sqrt[6]{(XY)^5}\right)^n + \\
&+ \left(0\cdot\sqrt[6]{(XY)^0} + 0\cdot\sqrt[6]{XY} + 1\cdot\sqrt[6]{(XY)^2} + 0\cdot\sqrt[6]{(XY)^3} + 1\cdot\sqrt[6]{(XY)^4} + 0\cdot\sqrt[6]{(XY)^5}\right)^n + \\
&+ \left(0\cdot\sqrt[6]{(XY)^0} + 0\cdot\sqrt[6]{XY} + 0\cdot\sqrt[6]{(XY)^2} + 1\cdot\sqrt[6]{(XY)^3} + 0\cdot\sqrt[6]{(XY)^4} + 0\cdot\sqrt[6]{(XY)^5}\right)^n + \\
&+ \left(Y\cdot\sqrt[6]{(XY)^0} + 0\cdot\sqrt[6]{XY} + 0\cdot\sqrt[6]{(XY)^2} + 0\cdot\sqrt[6]{(XY)^3} + 0\cdot\sqrt[6]{(XY)^4} + 0\cdot\sqrt[6]{(XY)^5}\right)^n \Bigg\}[X^pY^q] = \\
&= \Bigg\{ X^n + Y^n + \left(1 + XY + \sqrt[6]{XY} + 2\sqrt[6]{(XY)^2} + 2\sqrt[6]{(XY)^3} + 2\sqrt[6]{(XY)^4} + \sqrt[6]{(XY)^5}\right)^n + \\
&+ 2\cdot(XY)^{\frac{n}{2}} + \left(\sqrt[6]{(XY)^2} + \sqrt[6]{(XY)^4}\right)^n \Bigg\}[X^pY^q].
\end{aligned}
$$

3.3.2 $d = 4$

From table 5.5 and theorem 2.3.3 we get:

$$h^{p,q}\left(\widetilde{E_4^n/G_{4,n}}\right) = \left\{X^n+Y^n+\left(1+XY+2\sqrt[4]{XY}+3\sqrt[4]{(XY)^2}+2\sqrt[4]{(XY)^3}\right)^n+\left(\sqrt[4]{(XY)^2}\right)^n\right\}[X^pY^q].$$

3.3.3 $d = 3$

From table 5.7 and theorem 2.3.3 we get:

$$\begin{aligned} h^{p,q}\left(\widetilde{E_3^n/G_{3,n}}\right) &= \left\{X^n+Y^n+\left(1+XY+3\sqrt[3]{XY}+3\sqrt[3]{(XY)^2}\right)^n\right\}[X^pY^q] = \\ &= \left\{X^n+Y^n+\left(1+\sqrt[3]{XY}\right)^{3n}\right\}[X^pY^q]. \end{aligned}$$

3.3.4 $d = 2$

From table 5.9 and theorem 2.3.3 we get:

$$h^{p,q}\left(\widetilde{E_2^n/G_{2,n}}\right) = \left\{(X+Y)^n+\left(1+XY+4\sqrt{XY}\right)^n\right\}[X^pY^q]$$

Summarizing, we proved the following theorem:

Theorem 3.3.1. *The Hodge number* $h^{p,q}(X_{d,n}) = \left\{F_{X_{d,n}}(X,Y)\right\}[X^pY^q]$ *of the manifold* $X_{d,n}$ *is equal to*

$$\begin{cases} \left\{(X+Y)^n+\left(XY+4\sqrt{XY}+1\right)^n\right\}[X^pY^q] & \text{if } d=2, \\ \left\{X^n+Y^n+\left(1+\sqrt[3]{XY}\right)^{3n}\right\}[X^pY^q] & \text{if } d=3, \\ \left\{X^n+Y^n+\left(1+XY+2\sqrt[4]{XY}+3\sqrt[4]{(XY)^2}+2\sqrt[4]{(XY)^3}\right)^n+\left(\sqrt[4]{(XY)^2}\right)^n\right\}[X^pY^q] & \text{if } d=4, \\ \Big\{X^n+Y^n+\left(1+XY+\sqrt[6]{XY}+2\sqrt[6]{(XY)^2}+2\sqrt[6]{(XY)^3}+2\sqrt[6]{(XY)^4}+\sqrt[6]{(XY)^5}\right)^n+ & \\ +2\cdot(XY)^{\frac{n}{2}}+\left(\sqrt[6]{(XY)^2}+\sqrt[6]{(XY)^4}\right)^n\Big\}[X^pY^q] & \text{if } d=6. \end{cases}$$

Substituting appropriate roots of unity into the above formulas we get:

Corollary 3.3.2. *The Euler characteristic of manifold $X_{d,n}$ equals*

$$e\left(X_d^n\right) = \begin{cases} \frac{1}{2}(6^n + 3(-2)^n) & \text{if } d = 2, \\ \frac{1}{3}(8^n + 8(-1)^n) & \text{if } d = 3, \\ \frac{1}{4}(9^n + 3) + 3(-1)^n & \text{if } d = 4, \\ \frac{1}{6}(10^n + 3 \cdot 2^n + 8) + 4(-1)^n & \text{if } d = 6. \end{cases}$$

Remark 3.3.3. Theorem 3.3.1 yields

$$h^{1,n-1}(X_{2,n}) = h^1(\mathcal{T}_{X_{2,n}}) = n$$

for $d = 2$ and $n > 2$. Therefore, by the Tian-Todorov unobstructedness theorem the deformation space of $X_{2,n}$ has dimension n. On the other hand our construction involves n independent elliptic curves, so it depends on n parameters. Consequently the family $X_{2,n}$ is locally complete.

If $n > 2$ and $d = 3, 4, 6$ we get

$$h^{1,n-1}(X_{d,n}) = 0,$$

so the Calabi-Yau manifold $X_{d,n}$ is rigid.

3.3.5 Another method of computing Hodge numbers

In [Bur20] we used another approach for computing Hodge numbers of varieties $X_{d,n}$ based on a systematic study of orbits of the action of $G_{d,n}$. The method is very complex and we were not able to generalize it to the case of Calabi-Yau manifold other than elliptic curves.

For the sake of completeness we decided to reproduce a sketch of that proof.

Let E be an elliptic curve. Combining Künneth's formula with a standard induction argument we see that

$$h^{p,q}(E^n) = \binom{n}{p}\binom{n}{q}, \quad \text{for } 1 \le p, q \le n.$$

We begin with the following:

Lemma 3.3.4. *For any* $1 \leq p, q \leq n$, *the following equalities hold*

$$\dim H^{p,q}(E_d^n)^{G_{d,n}} = \begin{cases} \binom{n}{p} & \text{if } p = q \text{ or } p+q = n \text{ but } n \neq 2p, \\ 2\binom{n}{p} & \text{if } p = q \text{ and } p+q = n, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } d = 2.$$

$$\dim H^{p,q}(E_d^n)^{G_{d,n}} = \begin{cases} \binom{n}{p} & \text{if } p = q \text{ or } (p,q) \in \{(0,n),(n,0)\}, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } d = 3,4,6.$$

Proof. The Hodge vector space $H^{p,q}(E_d^n)$ is generated by differential forms of the following shape

$$\mathrm{d}z_{i_1} \wedge \mathrm{d}z_{i_2} \wedge \ldots \wedge \mathrm{d}z_{i_p} \wedge \mathrm{d}\overline{z_{j_1}} \wedge \mathrm{d}\overline{z_{j_2}} \wedge \ldots \wedge \mathrm{d}\overline{z_{j_q}}.$$

In the case of $d = 2$, we see that such (p,q)-form is $G_{2,n}$ invariant if and only if

- $p + q = n$ and $\{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q\} = \{1, 2, \ldots, n\}$ or
- $p = q$ and $\{i_1, i_2, \ldots, i_p\} = \{j_1, j_2, \ldots, j_q\}$.

Each of the cases provides $\binom{n}{p}$ choices.

Suppose that there exist indices $k \in \{i_1, i_2, \ldots, i_p\} \setminus \{j_1, j_2, \ldots, j_q\}$, and $l \in \{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q\}$ and without loss of generality assume that $k < l$. Then given (p,q)-form is not invariant under

$$\big(1, \ldots, \underbrace{-1}_{k-\text{th place}}, \ldots \underbrace{-1}_{l-\text{th place}}, \ldots 1\big).$$

In a similar way we prove the formula for $d = 3, 4, 6$. $\square$

$\mathbb{Z}/6\mathbb{Z}$ action

From the orbifold formula we get

$$H^{i,j}(X_{6,n}) := \bigoplus_{g \in G_{6,n}} \left(\bigoplus_{U \in \Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{G_{6,n}}, \tag{3.3.1}$$

where, $\Lambda(g)$ denotes the set of irreducible connected components of the set fixed by $g \in G_{6,n}$ and $\mathrm{age}(g)$ is the age of the matrix of linearised action of g near a point of U.

Now consider an element

$$g^{u,v,w,s,t} := \big(\underbrace{1, \ldots, 1}_{u}, \underbrace{2, \ldots, 2}_{v}, \underbrace{3, \ldots, 3}_{w}, \underbrace{4, \ldots, 4}_{s}, \underbrace{5, \ldots, 5}_{t}, \underbrace{0, \ldots, 0}_{\ell := n-u-v-w-s-t} \big) \in G_{6,n},$$

where $6 \mid u+2v+3w+4s+5t$, which corresponds to an automorphism of E_6^n such that the local action near a component of the fixed locus linearizes to

$$\big(\underbrace{\zeta_6,\dots,\zeta_6}_{u},\underbrace{\zeta_6^2,\dots,\zeta_6^2}_{v},\underbrace{\zeta_6^3,\dots,\zeta_6^3}_{w},\underbrace{\zeta_6^4,\dots,\zeta_6^4}_{s},\underbrace{\zeta_6^5,\dots,\zeta_6^5}_{t},\underbrace{1,\dots,1}_{\ell}\big).$$

Then $\operatorname{age}\left(g^{u,v,w,s,t}\right) = \frac{u+2v+3w+4s+5t}{6}$.

- The action of ϕ_6 and ϕ_6^5 have one fixed point: a, which stands for the infinity point of E_6.
- The action of ϕ_6^2 and ϕ_6^4 have three fixed points:

  $$a,\ b := (0,1),\ c := (0,-1)$$

  from which only a is invariant under ϕ_6 and the remaining two form a 2-cycle.
- The action of ϕ_6^3 has four fixed points:

  $$a,\ d := (1,0),\ e := (\zeta_3,0),\ f := (\zeta_3^2,0)$$

  from which only a is invariant under ϕ_6 and the remaining three form a 3-cycle.

If $\ell = n$ i.e. $(u,v,w,s,t) = (0,0,0,0,0)$, then $\mathrm{Fix}(g^{u,v,w,s,t}) = E_6^n$ and according to 3.3.4 the contribution to Poincaré polynomial corresponding to $g^{u,v,w,s,t}$ is equal to

$$X^n + Y^n + (1+XY)^n.$$

We shall study orbits of the action of $G_{6,n}$ on the set of irreducible components $\Lambda(g)$ of $\mathrm{Fix}(g)$ or equivalently on the finite set $F(g) := \mathrm{Fix}(\phi_6^{g_1}) \times \ldots \times \mathrm{Fix}(\phi_6^{g_{n-\ell}})$, where g_i denotes i-th coordinate of g.

If $u \neq 0$ or $t \neq 0$ or $\ell \neq 0$ then fixing $i \in \{1,2,\dots,n\}$ such that $g_i \in \{1,\zeta_6,\zeta_6^5\}$ and taking the element $h = (h_1,h_2,\dots,h_n) \in G_{6,n}$, where

$$h_k := \begin{cases} \zeta_6^5 & \text{if } k = i, \\ \zeta_6 & \text{if } k = j, \\ 1 & \text{if } k \notin \{i,j\}, \end{cases} \qquad \text{for } j \in \{1,2,\dots,n\} \setminus \{i\},$$

we see that each orbit of the action contains a unique element $x := (x_1,x_2,\dots,x_{n-\ell})$ with $x_i \in \{a,d\}$. The same holds true if ($v \neq 0$ or $s \neq 0$) and $w \neq 0$. Consequently the number of orbits equals 2^{v+w+s} unless $w = n$ or $v+s = n$.

On the other hand in the case $w = n$ each orbit of the action contains either a unique element $x = (x_1, x_2, \dots, x_n)$ with $x_i \in \{a, d\}$ or one of the following two elements: $(d, d, \dots, d, e)$ or $(d, d, \dots, d, f)$. Therefore we get $2^{v+w+s} + 2$ orbits in this situation.

Similar arguments show that in the case $v + s = n$ we get $2^{v+w+s} + 1$ orbits. As

$$\left(\bigoplus_{U \in \Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U) \right)^{G_{6,n}} = \left(\bigoplus_{U \in \Lambda(g)} H^{i-\mathrm{age}(g),\, j-\mathrm{age}(g)}(U)^{G_{6,\ell}} \right)^{G_{6,n}}$$

we get

$$\dim \left(\bigoplus_{U \in \Lambda(g)} H^{i,j}(U) \right)^{G_{6,n}} = \begin{cases} 1 & \text{if } (i,j) \in \{(n,0),(0,n)\}, u = 0, \\ 2^{v+w+s}\binom{\ell}{i} & \text{if } 0 \le i = j \le \ell, w \ne n, v+s \ne n, \\ 2^{v+w+s} + 2 & \text{if } w = n, i = j = 0, \\ 2^{v+w+s} + 1 & \text{if } v + s = n, i = j = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore the number $h^{p,q}(X_{6,n})$ is equal to the coefficient of X^pY^q in the polynomial:

$$X^n + Y^n + \sum_{u=0}^{n} \binom{n}{u} \sum_{v=0}^{n-u} \binom{n-u}{v} \sum_{w=0}^{n-u-v} \binom{n-u-v}{w} \sum_{s=0}^{n-u-v-w} \binom{n-u-v-w}{s} \times$$
$$\times \sum_{t=0}^{n-u-v-w-s} \binom{n-u-v-w-s}{t} \sum_{j=0}^{\ell} \binom{\ell}{j} \cdot 2^{v+w+s} (XY)^{j+\frac{u+2v+3w+4s+5t}{6}} + 2 \cdot (XY)^{\frac{n}{2}} +$$
$$+ \sum_{v=0}^{n} \binom{n}{v} (XY)^{\frac{1}{6}(2v+4(n-v))} = X^n + Y^n + \sum_{u=0}^{n} \binom{n}{u} \left(\sqrt[6]{XY}\right)^u \sum_{v=0}^{n-u} \binom{n-u}{v} \left(2\sqrt[6]{(XY)^2}\right)^v \times$$
$$\times \sum_{w=0}^{n-u-v} \binom{n-u-v}{w} \left(2\sqrt[6]{(XY)^2}\right)^w \sum_{s=0}^{n-u-v-w} \binom{n-u-v-w}{s} \left(2\sqrt[6]{(XY)^2}\right)^s \times$$
$$\times \sum_{t=0}^{n-u-v-w-s} \binom{n-u-v-w-s}{t} \left(\sqrt[6]{(XY)^2}\right)^t \sum_{j=0}^{\ell} \binom{\ell}{j} (XY)^j + 2 \cdot (XY)^{\frac{n}{2}} +$$
$$+ \left(\sqrt[6]{(XY)^2} + \sqrt[6]{(XY)^4} \right)^n =$$
$$= X^n + Y^n + \left(1 + XY + \sqrt[6]{XY} + 2\sqrt[6]{(XY)^2} + 2\sqrt[6]{(XY)^3} + 2\sqrt[6]{(XY)^4} + \sqrt[6]{(XY)^5}\right)^n +$$
$$+ 2 \cdot (XY)^{\frac{n}{2}} + \left(\sqrt[6]{(XY)^2} + \sqrt[6]{(XY)^4} \right)^n.$$

$\mathbb{Z}/4\mathbb{Z}$ action

Consider

$$g^{u,v,w} = \big(\underbrace{1, \dots, 1}_{u}, \underbrace{2, \dots, 2}_{v}, \underbrace{3, \dots, 3}_{w}, \underbrace{0, \dots, 0}_{n-u-v-w} \big) \in G_{4,n},$$

where $4 \mid u+2v+3w$. Then $\operatorname{age}(g^{u,v,w}) = \frac{u+2v+3w}{4}$.

Repeating the above arguments we get

$$\left\{F_{X_{4,n}}(X,Y)\right\}[X^pY^q] = X^n + Y^n + \frac{1}{2}\left\{\sum_{u=0}^{n}\sum_{v=0}^{n-u}\sum_{w=0}^{n-u-v}\sum_{j=0}^{n-u-v-w} 2^{v+w}\cdot 3^v \times\right.$$

$$\left.\times\binom{n}{u}\binom{n-u}{v}\binom{n-u-v}{w}\binom{n-u-v-w}{j}(XY)^{j+\frac{u+2v+3w}{4}} + \left(\sqrt[4]{(XY)^2}\right)^n\right\}[X^pY^q] =$$

$$= \left\{X^n + Y^n + \left(1 + XY + 2\sqrt[4]{XY} + 3\sqrt[4]{(XY)^2} + 2\sqrt[4]{(XY)^3}\right)^n + \left(\sqrt[4]{(XY)^2}\right)^n\right\}[X^pY^q].$$

$\mathbb{Z}/3\mathbb{Z}$ action

Take

$$g^{u,v} = (\underbrace{1,\dots,1}_{u},\underbrace{2,\dots,2}_{v},\underbrace{0,\dots,0}_{n-u-v}) \in G_{3,n},$$

where $3 \mid u+2v$. Then $\operatorname{age}(g^{u,v}) = \frac{u+2v}{3}$, hence from the orbifold formula we obtain

$$\left\{F_{X_{3,n}}(X,Y)\right\}[X^pY^q] =$$

$$= \left\{X^n + Y^n + \sum_{u=0}^{n}\sum_{v=0}^{n-u}\sum_{j=0}^{n-u-v} 3^u 3^v \binom{n}{u}\binom{n-u}{v}\binom{n-u-v}{j}(XY)^{j+\frac{u+2v}{3}}\right\}[X^pY^q] =$$

$$= \left\{X^n + Y^n + \left(1 + XY + 3\sqrt[3]{XY} + 3\sqrt[3]{(XY)^2}\right)^n\right\}[X^pY^q] =$$

$$= \left\{X^n + Y^n + \left(1 + \sqrt[3]{XY}\right)^{3n}\right\}[X^pY^q] =$$

$$= \left\{X^n + Y^n + \frac{1}{3}\left(\left(1+\sqrt[3]{XY}\right)^{3n} + \left(1+\zeta_3\sqrt[3]{XY}\right)^{3n} + \left(1+\zeta_3^2\sqrt[3]{XY}\right)^{3n}\right)\right\}[X^pY^q].$$

$\mathbb{Z}/2\mathbb{Z}$ action

From 3.3.4 we encode the Hodge numbers of fixed part of cohomology by the following generating polynomial function

$$\sum_{p,q=0}^{n} \dim H^{p,q}(E_2^n)^{G_{2,n}} X^pY^q = (1+XY)^n + (X+Y)^n.$$

Consider

$$g^u = (\underbrace{1,\dots,1}_{u},\underbrace{0,\dots,0}_{n-u}) \in G_{2,n},$$

an arbitrary element of $G_{2,n}$, where u is even. Then from the orbifold formula we have

$$\left\{F_{X_{2,n}}(X,Y)\right\}[X^pY^q] = \left\{(X+Y)^n + \sum_{u=0}^{n}\sum_{j=0}^{n-u} 4^u \binom{n}{u}\binom{n-u}{j}(XY)^{j+\frac{u}{2}}\right\}[X^pY^q] =$$

$$= \left\{(X+Y)^n + \left(1+XY+4\sqrt{XY}\right)^n\right\}[X^pY^q] =$$

$$= \left\{(X+Y)^n + \frac{1}{2}\left(\left(1+XY+4\sqrt{XY}\right)^n + \left(1+XY-4\sqrt{XY}\right)^n\right)\right\}[X^pY^q].$$

§ 3.4 Zeta functions

In this section we shall compute zeta functions of Calabi-Yau varieties $X_{d,n}$ using the method introduced in section 2.7. The zeta function of $X_{d,n}$ depends on an explicit model of the elliptic curve E_d over $\mathbb{Z}$, so we stick with the equations given in 3.1.

3.4.1 $d = 6$

For the elliptic curve E_6 with automorphism of order 6 the corresponding table 2.2 is equal to

$k \backslash j$	0	1	2	3	4	5
0	$\frac{1}{(1-T)(1-qT)}$	$1-\alpha_q T$	1	1	1	$1-\overline{\alpha_q} T$
1	$\frac{1}{1-T}$	1	1	1	1	1
2	$\frac{1}{(1-T)^2}$	1	1	$\frac{1}{1-T}$	1	1
3	$\frac{1}{(1-T)^2}$	1	$\frac{1}{1-T}$	1	$\frac{1}{1-T}$	1
4	$\frac{1}{(1-T)^2}$	1	1	$\frac{1}{1-T}$	1	1
5	$\frac{1}{1-T}$	1	1	1	1	1

Table 3.1: Local zeta functions of E_6

Here α_q and $\overline{\alpha_q}$ denote algebraic integers of modulus equal to $\sqrt{q}$.
Therefore by 2.7.1 we get

$$Z_q\left(X_{6,n}\right) = \left[\left(\frac{1}{(1-T)(1-q\cdot T)}\cdot\frac{1}{(1-\sqrt[6]{q}\cdot T)}\cdot\frac{1}{(1-\sqrt[6]{q^2}\cdot T)^2}\cdot\frac{1}{(1-\sqrt[6]{q^3}\cdot T)^2}\cdot\right.\right.$$

$$\left.\left.\cdot\frac{1}{(1-\sqrt[6]{q^4}\cdot T)^2}\cdot\frac{1}{(1-\sqrt[6]{q^5}\cdot T)}\right)\right]^{\otimes n}\times\left[(1-\alpha_q T)\right]^{\otimes n}\times\left[\left(\frac{1}{1-\sqrt[6]{q^3}\cdot T}\right)\right]^{\otimes n}\times$$

$$\times\left[\left(\frac{1}{1-\sqrt[6]{q^2}\cdot T}\cdot\frac{1}{1-\sqrt[6]{q^4}\cdot T}\right)\right]^{\otimes n}\times\left[\left(\frac{1}{1-\sqrt[6]{q^3}\cdot T}\right)\right]^{\otimes n}\times\left[(1-\overline{\alpha}_q\cdot T)\right]^{\otimes n} =$$

$$= \begin{cases} \dfrac{1}{\prod_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{6,n})}\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha}_q{}^nT\right)} & \text{if } 2 \mid n, \\[2ex] \dfrac{\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha}_q{}^nT\right)}{\prod_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{6,n})}} & \text{if } 2 \nmid n. \end{cases}$$

3.4.2 $d = 4$

For the elliptic curve E_4 with automorphism of order 4 the corresponding table 2.2 is equal to

$k \backslash j$	0	1	2	3
0	$\frac{1}{(1-T)(1-qT)}$	$1-\alpha_q T$	1	$1-\overline{\alpha}_q T$
1	$\frac{1}{(1-T)^2}$	1	1	1
2	$\frac{1}{(1-T)^3}$	1	$\frac{1}{1-T}$	1
3	$\frac{1}{(1-T)^2}$	1	1	1

Table 3.2: Local zeta functions of E_4

Therefore by 2.7.1 we get

$$
Z_q\left(X_{4,n}\right) = \left[\left(\frac{1}{(1-T)(1-q\cdot T)}\cdot\frac{1}{(1-\sqrt[4]{q}\cdot T)^2}\cdot\frac{1}{(1-\sqrt[4]{q^2}\cdot T)^3}\cdot\frac{1}{(1-\sqrt[4]{q^3}\cdot T)^2}\right)\right]^{\otimes n}\times
$$

$$
\times\left[(1-\alpha_q T)\right]^{\otimes n}\times\left[\left(\frac{1}{1-\sqrt[4]{q^2}\cdot T}\right)\right]^{\otimes n}\times\left[(1-\overline{\alpha_q}\cdot T)\right]^{\otimes n} =
$$

$$
= \begin{cases} \dfrac{1}{\prod\limits_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{4,n})}\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha_q}{}^nT\right)} & \text{if } 2 \mid n, \\[2ex] \dfrac{\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha_q}{}^nT\right)}{\prod\limits_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{4,n})}} & \text{if } 2 \nmid n. \end{cases}
$$

3.4.3 $d = 3$

For elliptic curve E_3 with automorphism of order 3 the corresponding table 2.2 is equal to

$k \backslash j$	0	1	2
0	$\frac{1}{(1-T)(1-qT)}$	$1-\alpha_q T$	$1-\overline{\alpha_q}T$
1	$\frac{1}{(1-T)^3}$	1	1
2	$\frac{1}{(1-T)^3}$	1	1

Table 3.3: Local zeta functions of E_3

Therefore by 2.7.1 we get

$$Z_q\left(X_{3,n}\right) = \left[\left(\frac{1}{(1-T)(1-q\cdot T)}\cdot\frac{1}{(1-\sqrt[3]{q}\cdot T)^3}\cdot\frac{1}{(1-\sqrt[3]{q^2}\cdot T)^3}\right]^{\otimes n}\times$$
$$\times\left[(1-\alpha_q T)\right]^{\otimes n}\times\left[(1-\overline{\alpha_q}\cdot T)\right]^{\otimes n} =$$

$$= \begin{cases} \dfrac{1}{\prod_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{3,n})}\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha_q}{}^nT\right)} & \text{if } 2 \mid n, \\ \dfrac{\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha_q}{}^nT\right)}{\prod_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{3,n})}} & \text{if } 2 \nmid n. \end{cases}$$

Summarizing, we proved the following theorem

Theorem 3.4.1. *The zeta function of the manifold $X_{d,n}$ is equal to*

$$\begin{cases} \dfrac{1}{\prod_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{d,n})}\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha_q}{}^nT\right)} & \text{if } 2 \mid n, \\ \dfrac{\left(1-\alpha_q{}^nT\right)\left(1-\overline{\alpha_q}{}^nT\right)}{\prod_{i=0}^{n}(1-q^iT)^{h^{i,i}(X_{d,n})}} & \text{if } 2 \nmid n. \end{cases}$$

3.4.4 $d = 2$

In this case E is an arbitrary elliptic curve so we cannot describe α_q and $\overline{\alpha_q}$ in general. However $\alpha_q + \overline{\alpha_q}$ is the trace a_q of Frobenius morphism Frob_q^* on $H^{**}(E)$. Now $\alpha_q^n + \overline{\alpha_q}^n$ can be computed as a polynomial in a_q.

The corresponding table 2.2 which will be discuss in Chapter 5 is equal to

$k \backslash j$	0	1
0	$\dfrac{1}{(1-T)(1-qT)}$	$1 - a_q T + qT^2$
1	$\dfrac{1}{(1-T)^4}$, $\dfrac{1}{(1-T)^3(1+T)}$, $\dfrac{1}{(1-T)^2(1+T+T^2)}$	1

Table 3.4: $Z_{E,k,j}(T)$

Here we have three possible rational functions $Z_{E,1,0}$ depending on number of fixed point of an involution on E defined over $\mathbb{Q}$. We give only formulas for $n = 2, 3$ since the general one is too complicated.

When $Z_{E,1,0} = \dfrac{1}{(1-T)^4}$ by 2.7.1 we get

n	$Z_q\left(\widetilde{E^n/\mathbb{Z}_2^{n-1}}\right)$
2	$\dfrac{1}{(1-T)(1-qT)^{20}(1+2qT-a_q^2T+q^2T^2)(1-q^2T)}$
3	$\dfrac{(1-qa_qT+q^3T^2)^3(1+3qa_qT-a_q^3T+q^3T^2)}{(1-T)(1-qT)^{51}\left(1-q^2T\right)^{51}\left(1-q^3T\right)}$

Table 3.5: Zeta function of $X_{2,2}$, $X_{2,3}$

When $Z_{E,1,0} = \dfrac{1}{(1-T)^3(1+T)}$ by 2.7.1 we get

n	$Z_q\left(\widetilde{E^n/\mathbb{Z}_2^{n-1}}\right)$
2	$\dfrac{1}{(1-T)(1-qT)^{14}(1+qT)^6(1+2qT-a_q^2T+q^2T^2)(1-q^2T)}$
3	$\dfrac{(1-qa_qT+q^3T^2)^3(1+3qa_qT-a_q^3T+q^3T^2)}{(1-T)(1-qT)^{33}(1+qT)^{18}\left(1-q^2T\right)^{33}\left(1+q^2T\right)^{18}\left(1-q^3T\right)}$

Table 3.6: Zeta function of $X_{2,2}$, $X_{2,3}$

When $Z_{E,1,0} = \dfrac{1}{(1-T)^2(1+T+T^2)}$ by 2.7.1 we get

n	$Z_q\left(\widetilde{E^n/\mathbb{Z}_2^{n-1}}\right)$
2	$\dfrac{1}{(1-T)(1-qT)^{10}(1+qT+q^2T^2)^5(1+2qT-a_q^2T+q^2T^2)(1-q^2T)}$
3	$\dfrac{(1-qa_qT+q^3T^2)^3(1+3qa_qT-a_q^3T+q^3T^2)}{(1-T)(1-qT)^{21}\left(1+qT+q^2T^2\right)^{15}\left(1-q^2T\right)^{21}\left(1+q^2T+q^4T^2\right)^{15}\left(1-q^3T\right)}$

Table 3.7: Zeta function of $X_{2,2}$, $X_{2,3}$

Remark 3.4.2. In [CH07] authors computed L-function of varieties $X_{d,n}$ for $d = 2, 3$:

$$L(X_{d,n}, s) = \begin{cases} L(g_{n+1}, s), & \text{for } d = 3, \text{ where } g_{n+1} \text{ is a weight } n+1 \text{ cusp form with CM in } \mathbb{Q}(\sqrt{-3}) \\ L(g_{n+1}, s), & \text{for } d = 4, \text{ where } g_{n+1} \text{ is a weight } n+1 \text{ cusp form with CM in } \mathbb{Q}(i) \end{cases},$$

Chapter 4

Zariski varieties

In this chapter we discuss rationality problems in algebraic geometry. We shall use Cynk-Hulek construction in order to extend result of Katsura-Schuett to obtain higher dimensional Calabi-Yau manifolds, which are Zariski varieties.

§ 4.1 Rationality problems in algebraic geometry

Let us begin with the following definition which is crucial for the present section:

Definition 4.1.1. A variety X is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, X is *rational* if there exists a birational map $\mathbb{P}^n \overset{\sim}{\dashrightarrow} X$.

Of course, the definition of unirationality of variety can be stated in terms of field extensions i.e. X is unirational if and only if its functional field $\mathbb{C}(X)$ has a finite extension, which is a purely transcendental extension over $\mathbb{C}$.

The classical problem of Lüroth asks whether every unirational variety is rational. Therefore Lüroth's problems asks whether any extension of $\mathbb{C}$ contained in $\mathbb{C}(t_1, t_2, \dots, t_n)$ is purely transcendental.

For curves it was proven by Lüroth ([Lür75]). In dimension 2 Castelnuovo found a criterion for surface to being rational i.e. smooth, projective surface S is rational if and only if $p_2(S) = q(S) = 0$, where $p_2(S) := h^0(X, \omega_S^2)$ and $q(S) := h^1(S, \mathcal{O}_S)$. If we have dominant rational map $\mathbb{P}^2 \dashrightarrow S$, then of course $p_2(S) = q(S) = 0$ and so S is rational, hence Lüroth's question for surfaces remains true.

In the beginning of 20 century first counter-example to Lüroth's problem in dimension 3 was constructed by Enriques ([Enr12]) as a complete intersection of quadric and cubic in

$\mathbb{P}^5$. After 50 years three different counter-examples appeared, together with new methods. Precisely:

- Clemens-Griffiths in [CG72] proved that smooth cubic threefolds in $\mathbb{P}^4$ (which are in fact unirational) are not rational. The idea was to show that intermediate jacobian of such varieties is not a Jacobian.

- Iskovskikh-Manin in [IM72] proved that a smooth quartic threefolds in $\mathbb{P}^4$ is not rational. Since there exists known examples of unirational quartic threefolds, this result leads to another counter-example to Lüroth's problem. The idea was to compare birational automorphisms group of quartic threefold (which is finite) with corresponding automorphisms group of $\mathbb{P}^3$ (in fact very large).

- Artin-Mumford in [AM72] proved that double cover of $\mathbb{P}^3$ branched along a quartic surface in $\mathbb{P}^3$ with 10 nodes is unirational and not rational. The idea was to show that torsion of third cohomology group of this variety is non-trivial, leading to new birational invariant.

We refer to book [Bea+16] for a very good and complete exposition of the past and modern approaches in these problems.

In positive characteristic Lüroth's theorem is still true (over algebraically closed field) in one dimensional case, but this is no longer true for surfaces as was first shown by Zariski ([Zar58]).

Definition 4.1.2. An algebraic (non-rational) surface S, over algebraically closed field of characteristic p is called a *Zariski surface* if there exists a purely inseparable dominant rational map $\mathbb{P}^2 \dashrightarrow X$ of degree p.

The above definition allows us to thinking about Zariski surfaces as a first non-rational and unirational surfaces over algebraically closed field of characteristic p.

§ 4.2 Zariski K3 surfaces

From [Shi74] it follows that Zariski surfaces S are automatically *supersingular* i.e.

$$\text{second Betti number of } S = \text{Picard number of } S.$$

This implication leads to the following question, originally posed by Shioda in [Shi77]:

Question 4.2.1. Is any supersingular surface a Zariski surface?

There are some partial results in this question concerning supersingular $K3$ surfaces. Unirationality of supersingular $K3$ surfaces is known in characteristic 2, this results was proven in [RŠ78] by Rudakov and Shafarevich. Moreover Artin and Shioda in papers [Art74], [Shi74] constructed supersingular $K3$ surfaces which are unirational. Today, all these results are direct consequence of the result of Liedtke (cf. [Lie15]), who proved that all supersingular $K3$ surfaces are unirational.

Lack of general results and constructions in arbitrary large characteristics indicates that the answer to 4.2.1 in full generality might be difficult. Recently Katsura and Shütt in [KS20] gave partial answer to restricted question:

Question 4.2.2. Is any supersingular Kummer surface is a Zariski surface?

Katsura and Schütt constructed first examples of Zariski $K3$ surfaces using the classical Kummer construction in dimension 2. The crucial part of their idea was a special endomorphism of supersingular elliptic curves admitting automorphisms of order 3 and 4. They proved the following theorem:

Theorem 4.2.3. *Let $p > 2$ be a prime such that $p \not\equiv 1 \pmod{12}$. Then any supersingular Kummer surface in characteristic p is a Zariski surface.*

In particular they constructed first example of $K3$ surface which is a Zariski variety.

§ 4.3 Zariski Calabi-Yau varieties

In this section we extend the argument given in [KS20] to obtain higher dimensional Calabi-Yau manifolds, which are Zariski varieties.

Definition 4.3.1. An algebraic (non-rational) variety X of dimension n, over algebraically closed field of characteristic p is called a *Zariski variety* if there exists a purely inseparable dominant rational map $\mathbb{P}^n \dashrightarrow X$ of degree p.

4.3.1 $\mathbb{Z}/3\mathbb{Z}$ action

Let $E_{3,i}$ be the elliptic curve given by the equation $y_i^2 + y_i = x_i^3$, for $i \in \{1, 2, \ldots, n\}$ with the ζ_3 action $\tau_3 : (x, y) \mapsto (\zeta_3 x, y)$ and consider groups

$$F_i := \left\langle (\tau_3, 1, \ldots, 1, \tau_3^i),\ (1, \tau_3, 1, \ldots, 1, \tau_3^i),\ \ldots,\ (1, \ldots, 1, \tau_3, \tau_3^i) \right\rangle \simeq \mathbb{Z}_3^{n-1} \simeq G_{3,n},$$

for $i = 1, 2$.

Lemma 4.3.2. *The quotient variety* $Z_{3,n} := {E_{3,1} \times E_{3,2} \times \ldots \times E_{3,n}}\big/_{F_1}$ *is rational.*

Proof. The monomial $x_1^{i_1} x_2^{i_2} \cdot \ldots \cdot x_n^{i_n} y_1^{j_1} y_2^{j_2} \cdot \ldots \cdot y_n^{j_n}$ is invariant under F_1 iff $3 \mid i_n + i_k$ for $1 \leq k < n$, thus

$$\mathbb{C}[Z_{3,n}] \simeq \mathbb{C}[y_1, y_2, \ldots, y_n, x_1 x_2 \ldots x_{n-1} x_n^2, x_1^2 x_2^2 \ldots x_{n-1}^2 x_n].$$

Now let $z := \dfrac{x_1 x_2 \ldots x_{n-1}}{x_n}$ and observe that

$$\mathbb{C}(Z_{3,n}) = \mathbb{C}(y_1, y_2, \ldots, y_n, z),$$

since

$$x_1 x_2 \ldots x_{n-1} x_n^2 = z(y_n^2 + y_n) \quad \text{and} \quad x_1^2 x_2^2 \ldots x_{n-1}^2 x_n = z^2(y_n^2 + y_n).$$

Moreover we have the following relation

$$z^3 = \left(\frac{x_1 x_2 \ldots x_{n-1}}{x_n} \right)^3 = \frac{(y_1^2 + y_1)(y_2^2 + y_2) \ldots (y_{n-1}^2 + y_{n-1})}{y_n^2 + y_n}$$

or equivalently

$$(y_1^2 + y_1)(y_2^2 + y_2) \cdot \ldots \cdot (y_{n-1}^2 + y_{n-1}) = z^3 (y_n^2 + y_n).$$

Taking $\alpha := \dfrac{y_n}{y_{n-1}}$, we get the equation

$$(y_1^2 + y_1)(y_2^2 + y_2) \cdot \ldots \cdot (y_{n-2}^2 + y_{n-2})(y_{n-1} + 1) = z^3 \alpha(\alpha y_{n-1} + 1),$$

from which we can compute y_{n-1} and $y_n = \alpha y_{n-1}$ as rational functions in y_1, y_2, $\ldots$, y_{n-2}, z, α. Hence the variety $Z_{3,n}$ is rational. $\square$

Now, consider a prime number $p \equiv 2 \pmod 3$ and the supersingular elliptic curve E_3 over a field k, such that $\zeta_3 \in k$ and char $k = p$, defined by equation $y^2 + y = x^3$, and with the ζ_3 action $\tau_3 \colon (x, y) \mapsto (\zeta_3 x, y)$. The endomorphism ring of E_3 may be represented as

$$\mathrm{End}(E_3) = \mathbb{Z} \oplus \mathbb{Z}F \oplus \mathbb{Z}\tau_3 \oplus \mathbb{Z}\frac{(1 + F)(2 + \tau_3)}{3},$$

where F is a Frobenius morphism of E_3, with the relation $F\tau_3 = \tau_3^2 F$ (cf. [Kat87]).

Theorem 4.3.3. *The Calabi-Yau manifold* $\widetilde{{E_3^n}\big/_{F_2}} = X_{3,n}$ *is a Zariski manifold.*

Proof. Commutativity of the following diagram

$$\begin{tikzcd} E_3^n \arrow[r, "F_1"] \arrow[d, "1\times\ldots\times 1\times F"'] \arrow[dr, phantom, "\circlearrowleft"] & E_3^n \arrow[d, "1\times\ldots\times 1\times F"] \\ E_3^n \arrow[r, "F_2"'] & E_3^n \end{tikzcd}$$

leads to purely inseparable rational map $E_3^n/_{F_1} \longrightarrow E_3^n/_{F_2}$ of degree p. Since $E_3^n/_{F_1}$ by 4.3.2 is a rational variety, the theorem follows. $\square$

4.3.2 $\mathbb{Z}/4\mathbb{Z}$ action

Let $E_{4,i}$ be the elliptic curve given by the equation $y_i^2 = x_i^3 - x_i$, for $i \in \{1, 2, \ldots, n\}$ with the ζ_4 action $\tau_4 : (x, y) \mapsto (-x, iy)$ and consider groups

$$H_i := \left\langle (\tau_4, 1, \ldots, 1, \tau_4^i),\ (1, \tau_4, 1, \ldots, 1, \tau_4^i),\ \ldots,\ (1, \ldots, 1, \tau_4, \tau_4^i) \right\rangle \simeq \mathbb{Z}_4^{n-1} \simeq G_{4,n},$$

for $i = 1, 3$.

As in the previous section the following lemma holds

Lemma 4.3.4. *The quotient variety* $Z_{4,n} := E_{4,1} \times E_{4,2} \times \ldots \times E_{4,n}/_{H_1}$ *is rational.*

Proof. The monomial $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} y_1^{b_1} y_2^{b_2} \ldots y_n^{b_n}$ is invariant under H_1 iff either

(1) $2 \mid a_i + a_n$ and $4 \mid b_i + b_n$ for $0 \le i < n$, which are generated by

$$y_1^4,\ y_2^4,\ \ldots,\ y_n^4,\ y_1 y_2 \cdots y_{n-1} y_n^3,\ y_1^3 y_2^3 \cdots y_{n-1}^3 y_n,\ x_1^2,\ x_2^2,\ \ldots,\ x_n^2,\ x_1 x_2 \ldots x_{n-1} x_n$$

or

(2) $2 \nmid a_i + a_n$ and $b_i + b_n \equiv 2 \pmod 4$ for $0 \le i < n$, which are generated by

$$\begin{gathered} y_1^2 y_2^2 \cdots y_{n-1}^2 x_1 x_2 \ldots x_{n-1},\quad y_1^2 y_2^2 \cdots y_{n-1}^2 x_n,\quad y_1 y_2 \cdots y_n x_1 x_2 \ldots x_{n-1} \\ y_1 y_2 \cdots y_n x_n,\quad y_n^2 x_1 x_2 \ldots x_{n-1},\quad y_n^2 x_n,\quad y_1^3 y_2^3 \cdots y_{n-1}^3 y_n^3 x_1 x_2 \cdots x_{n-1} \\ y_1^3 y_2^3 \cdots y_{n-1}^3 y_n^3 x_n,\quad y_1^2 y_2^2 \cdots y_{n-1}^2 y_n^4 x_n,\quad y_1^2 y_2^2 \cdots y_{n-1}^2 y_n^4 x_1 x_2 \ldots x_{n-1}. \end{gathered}$$

Let us take $t_i := x_i^2$, $z_1 = \dfrac{x_1 x_2 \ldots x_{n-1}}{x_n}$ and $z_2 = \dfrac{y_1 y_2 \cdots y_{n-1}}{y_n}$ and observe that

$$\mathbb{C}(Z_{4,n}) = \mathbb{C}\left(t_1, t_2, \ldots, t_n, z_2\right),$$

indeed it follows from identities

$$
\begin{aligned}
&y_i^4 = t_i^3 - 2t_i^2 + t_i, \;\; y_1y_2\dots y_{n-1}y_n^3 = z_2y_n^4, \;\; y_1^3y_2^3\dots y_{n-1}^3y_n = z_2^3y_n^4,\\
&x_1x_2\dots x_{n-1}x_n = z_1t_n, \;\; y_n^2x_n = t_n(t_n-1), \;\; y_1^2y_2^2\dots y_{n-1}^2x_1x_2\dots x_{n-1} = z_1z_2^2\cdot(y_n^2x_n),\\
&y_1^2y_2^2\dots y_{n-1}^2x_n = z_2^2\cdot(y_n^2x_n), \;\; y_1y_2\dots y_nx_1x_2\dots x_{n-1} = z_2\cdot(y_n^2x_n)\cdot z_1,\\
&y_1y_2\dots y_nx_n = z_2\cdot(y_n^2x_n), \;\; y_n^2x_1x_2\dots x_{n-1} = (y_n^2x_n)\cdot z_1, \;\; y_1^3y_2^3\dots y_{n-1}^3y_n^3x_n = z_2^3y_n^4\cdot(y_n^2x_n),\\
&y_1^3y_2^3\dots y_{n-1}^3y_n^3x_1x_2\dots x_{n-1} = z_2^3\cdot(y_n^2x_n)\cdot y_n^4z_1, \;\; y_1^2y_2^2\dots y_{n-1}^2y_n^4x_n = z_2^2\cdot(y_2^2x_n)\cdot y_n^4,\\
&y_1^2y_2^2\dots y_{n-1}^2y_n^4x_1x_2\dots x_{n-1} = z_2^2\cdot(y_2^2x_n)\cdot y_n^4z_1, \;\; z_1 = \frac{z_2^2(t_n-1)}{(t_1-1)(t_2-1)\dots(t_{n-1}-1)}.
\end{aligned}
$$

The variety $Z_{4,n}$ may be defined by the equation

$$z_2^4t_n(t_n-1)^2 = t_1t_2\dots t_{n-1}(t_1-1)^2(t_2-1)^2\dots(t_{n-1}-1)^2.$$

Taking $\alpha := \dfrac{t_n-1}{t_{n-1}-1}$, we get the equation

$$z_2^4\alpha^2(\alpha(t_{n-1}-1)+1) = t_1t_2\dots t_{n-1}(t_1-1)^2(t_2-1)^2\dots(t_{n-2}-1)^2,$$

which is linear in t_{n-1}, so

$$\mathbb{C}(Z_{4,n}) = \mathbb{C}(t_1, t_2, \dots, t_{n-2}, z, \alpha).$$

□

Now, we assume $p \equiv 3 \pmod 4$. Consider the supersingular elliptic curve E_4 defined by the equation $y^2 = x^3 - x$ with order 4 automorphism $\tau_4(x,y) = (-x, iy)$. The endomorphism ring of E_4 may be represented as

$$\mathrm{End}(E_4) = \mathbb{Z} \oplus \mathbb{Z}\tau_4 \oplus \mathbb{Z}\left(\frac{1+F}{2}\right) \oplus \mathbb{Z}\tau_4\left(\frac{1+F}{2}\right)$$

with the relation $F\tau_4 = \tau_4^3F$ (cf. [Kat87]).

Theorem 4.3.5. *The Calabi-Yau manifold* $\widetilde{E_4^n/H_3} = X_{4,n}$ *is a Zariski manifold.*

Proof. The commutativity of the diagram

$$\begin{tikzcd}
E_4^n \arrow[r, "H_1"] \arrow[d, "1\times\dots\times1\times F"'] \arrow[dr, phantom, "\circlearrowleft"] & E_4^n \arrow[d, "1\times\dots\times1\times F"] \\
E_4^n \arrow[r, "H_3"'] & E_4^n
\end{tikzcd}$$

leads to purely inseparable rational map ${E_4^n}/{H_1} \longrightarrow {E_4^n}/{H_3}$ of degree p. Since ${E_4^n}/{H_1}$ by 4.3.2 is a rational variety, the theorem follows.

$\square$

4.3.3 $\mathbb{Z}/6\mathbb{Z}$ action

Let $E_{6,i}$ be the elliptic curve given by the equation $y_i^2 + y_i = x_i^3$, for $i \in \{1, 2, \dots, n\}$ with the ζ_6 action $\tau_6 : (x, y) \mapsto (\zeta_3 x, -y - 1)$ and consider groups

$$J_i := \left\langle (\tau_6, 1, \dots, 1, \tau_6^i),\ (1, \tau_6, 1, \dots, 1, \tau_6^i),\ \dots,\ (1, \dots, 1, \tau_6, \tau_6^i) \right\rangle \simeq \mathbb{Z}_6^{n-1} \simeq G_{6,n},$$

for $i = 1, 5$.

Lemma 4.3.6. *The quotient variety* $Z_{6,n} := {E_{6,1} \times E_{6,2} \times \ldots \times E_{6,n}}/{J_1}$ *is rational.*

Proof. The monomials $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n} y_1^{b_1} y_2^{b_2} \dots y_n^{b_n}$ are invariant under J_1 iff $b_i = 0$ for $1 \le i \le n$ and for any $1 \le i < n$: $3 \mid a_i + a_n$. Consequently, these monomials are generated by:

$$x_1^3,\ x_2^3,\ \dots,\ x_n^3,\ x_1 x_2 \dots x_{n-1} x_n^2,\ x_1^2 x_2^2 \dots x_{n-1}^2 x_n.$$

Now let $z := \dfrac{x_1 x_2 \dots x_{n-1}}{x_n}$ and observe that

$$\mathbb{C}(Z_{6,n}) = \mathbb{C}(y_1, y_2, \dots, y_n, z),$$

since

$$x_i^3 = y_i^2 + y_i, \quad x_1 x_2 \dots x_{n-1} x_n^2 = z(y_n^2 + y_n) \quad \text{and} \quad x_1^2 x_2^2 \dots x_{n-1}^2 x_n = z^2(y_n^2 + y_n).$$

Moreover we have the following relation

$$z^3 = \left(\frac{x_1 x_2 \dots x_{n-1}}{x_n} \right)^3 = \frac{(y_1^2 + y_1)(y_2^2 + y_2) \dots (y_{n-1}^2 + y_{n-1})}{y_n^2 + y_n}$$

or equivalently

$$(y_1^2 + y_1)(y_2^2 + y_2) \cdot \ldots \cdot (y_{n-1}^2 + y_{n-1}) = z^3 (y_n^2 + y_n).$$

Taking $\alpha := \dfrac{y_n}{y_{n-1}}$, we get the equation

$$(y_1^2 + y_1)(y_2^2 + y_2) \cdot \ldots \cdot (y_{n-2}^2 + y_{n-2})(y_{n-1} + 1) = z^3 \alpha (\alpha y_{n-1} + 1),$$

from which we can compute y_{n-1} and $y_n = \alpha y_{n-1}$ as rational functions in $y_1, y_2, \dots, y_{n-2}$, z, α. Hence the variety $Z_{6,n}$ is rational.

$\square$

In that case we use elliptic curve with the same equation as in the case of the action by $\mathbb{Z}/3\mathbb{Z}$. Take a prime number $p \equiv 2 \pmod 3$ and the supersingular elliptic curve E_6 over a field k, such that $\zeta_6 \in k$ and char $k = p$, defined by equation $y^2 + y = x^3$, and with the ζ_6 action $\tau_6 : (x, y) \mapsto (\zeta_3 x, -1 - y)$. The endomorphism ring of E_6 may be represented as

$$\mathrm{End}(E_6) = \mathbb{Z} \oplus \mathbb{Z}F \oplus \mathbb{Z}\tau_6 \oplus \mathbb{Z}\frac{(1+F)(2-\tau_6)}{3},$$

where F is a Frobenius morphism of E_6, with the relation $F\tau_6 = \tau_6^5 F$.

Theorem 4.3.7. *The Calabi-Yau manifold* $\widetilde{E_6^n/J_5} = X_{6,n}$ *is a Zariski manifold.*

Proof. The commutativity of the diagram

$$\begin{tikzcd} E_6^n \arrow[r, "J_1"] \arrow[d, "1\times\ldots\times1\times F"'] \arrow[dr, phantom, "\circlearrowright"] & E_6^n \arrow[d, "1\times\ldots\times1\times F"] \\ E_6^n \arrow[r, "J_5"'] & E_6^n \end{tikzcd}$$

leads to purely inseparable rational map $E_6^n/J_1 \longrightarrow E_6^n/J_5$ of degree p. Since E_6^n/J_1 by 4.3.2 is a rational variety, the theorem follows. $\square$

Directly from the previous sections the following corollaries hold:

Corollary 4.3.8. *In any odd characteristic* $p \not\equiv 1 \pmod{12}$ *there exists a Zariski Calabi-Yau manifold of arbitrary dimension.*

Corollary 4.3.9. *In any odd characteristic* $p \not\equiv 1 \pmod{12}$ *there exists a unirational Calabi-Yau manifold of arbitrary dimension.*

Remark 4.3.10. According to our knowledge constructed varieties are the first example of higher dimensional (at least 3) Calabi-Yau manifolds with Zariski property. However there are unirational Calabi-Yau manifolds of arbitrary dimension in many positive characteristics (in fact in characteristic 0, a Calabi-Yau variety cannot be unirational, cf. [MM86]) i.e.

- Hirokado's construction of Calabi-Yau threefold obtained as a quotient of $\mathbb{P}^3$ by a p-closed rational vector field in characteristic $p = 3$ (see [Hir99]). In fact Hirokado's Calabi-Yau threefold is a non-liftable Calabi-Yau threefold in characteristic 3.

- In papers [HIS07], [HIS08] authors adopted Schoen's construction of Calabi-Yau threefolds using elliptic surfaces ([Sch88]) to obtain unirational Calabi-Yau threefolds in characteristic 3.

- Let X_m^r denotes Fermat variety of dimension r and degree m i.e. smooth hypersurface in $\mathbb{P}^{r+1}$ given by the following equation:

  $$X_0^m + X_1^m + \ldots + X_{r+1}^m = 0.$$

  In [SK79] it is proven that if $p^\ell \equiv -1 \pmod{m}$ for some integer ℓ, and for any even positive integer r, the Fermat variety X_m^r is unirational in characteristic p.

  Therefore, we may take arbitrary even positive integer m, put $r := m-2$ and consider prime p which is $-1 \pmod{m}$ (there are infinitely many such primes). Then X_{m-2}^m is unirational Calabi-Yau variety in characteristic p. See also [Shi92] for a construction of unirational complete intersections of Fermat's hypersurfaces in positive characteristic.

Chapter 5

Calabi-Yau varieties of Borcea-Voisin type

One of the first important achievements in the theory of Calabi-Yau threefolds and particularly in Mirror Symmetry Conjecture, was the construction given independently by C. Borcea ([Bor97]) and C. Voisin ([Voi93]). They constructed families of Calabi-Yau threefolds using a non-symplectic involutions of $K3$ surfaces and elliptic curves. Moreover C. Voisin gave a construction of explicit mirror maps.

The Borcea-Voisin construction is actually similar to the one given by C. Vafa and E. Witten in [VW95]. They divided a product of three tori by a group of automorphisms preserving the volume form. This approach gave rise to abstract physical models studied by L. Dixon, J. Harvey, C. Vafa, E. Witten in [Dix+85; Dix+86]. Similar constructions i.e. quotients of products of tori by a finite group were classified by J. Dillies, R. Donagi, A. E. Faraggi an K. Wendland in [Dil07; DF04; DW09].

There are many generalizations of the above constructions. The first idea is to allow automorphisms of higher order. In [Roh10] Rohde constructed Calabi-Yau threefolds by taking a quotient of a product of an elliptic curve and a $K3$ surface by an automorphism of order 3, fixing only points or rational curves on the $K3$ surface. A. Molnar in his PhD thesis ([Mol15]) found another groups acting on a product of three elliptic curves and studied modularity of the resulting quotients. S. Cynk and K. Hulek study examples of threefolds (and higher dimensional varieties) using involutions and higher order automorphisms (see [CH07]); they also proved their modularity. Finally [CG16] A. Cattaneo and A. Garbagnati used purely non-symplectic automorphisms of order 3, 4 and 6 to generalized the Borcea-Voisin construction.

Another possibility is to take a quotient of a product of two $K3$ surfaces by a finite group. Such fourfolds were studied by J. Dillies in [Dil12a]. F. Reidegald divided $S \times \mathbb{P}^1$, where

S is a $K3$ surface, by a cyclic group of order 3. He also found a desingularization of such quotients (see [Rei15]).

Our attempt to generalize Borcea-Voisin construction is to allow more elliptic curves as a factor and make construction similar to Cynk-Hulek construction in chapter 2. At the same time we increase the dimension of the resulting variety.

In this chapter we briefly recall the original Borcea-Voisin construction and its generalization given by A. Cattaneo and A. Garbagnati. Then we shall give higher dimensional generalisation and compute Hodge numbers of resulting varieties using the orbifold cohomology formula and the stringy Euler characteristic.

§ 5.1 Borcea-Voisin construction

One of the many reasons behind the interest in non-symplectic automorphisms of $K3$ surfaces is the mirror symmetry construction of C. Borcea ([Bor97]) and C. Voisin ([Voi93]). They independently constructed a family of Calabi-Yau threefolds using a non-symplectic involutions of $K3$ surfaces and elliptic curves. Moreover C. Voisin gave a construction of explicit mirror maps.

Theorem 5.1.1 ([Bor97; Voi93]). *Let E be an elliptic curve with an involution α_E which does not preserve ω_E. Let S be a $K3$ surface with a non-symplectic involution α_S. Then any crepant resolution of the variety $E \times S /_{\alpha_E \times \alpha_S}$ is a Calabi-Yau manifold with*

$$h^{1,1} = 11 + 5N - N' \quad \textit{and} \quad h^{2,1} = 11 + 5N' - N,$$

where N is the number of curves in $\mathrm{Fix}(\alpha_S)$ *and N' is the sum of their genera.*

C. Borcea considered any two varieties (X, σ_X) and (Y, σ_Y) of the Calabi-Yau type (i.e. $K_X = K_Y = 0$) with involutions σ_X and σ_Y and made the blow up of $X \times Y$ along $\mathrm{Fix}(\sigma_X) \times \mathrm{Fix}(\sigma_Y)$ which produced a crepant desingularisation of $X \times Y /_{\sigma_X \times \sigma_Y}$.

From Nikulin's classification ([Nik87]) it follows that for any $K3$ surface S with non-symplectic involution α_S which fixes N curves with sum of genera equal to N', there exists a complementary surface S' and its non-symplectic involution α'_S with N' fixed curves with sum of genera equal to N. Thus we have the following corollary:

Corollary 5.1.2. *The pair* $\widetilde{E \times S/_{\alpha_E \times \alpha_S}}$, $\widetilde{E \times S'/_{\alpha_E \times \alpha_{S'}}}$ *is a mirror pair i.e.*

$$h^{1,1}\left(\widetilde{E \times S/_{\alpha_E \times \alpha_S}}\right) = h^{2,1}\left(\widetilde{E \times S'/_{\alpha_E \times \alpha_{S'}}}\right) \quad \text{and}$$
$$h^{2,1}\left(\widetilde{E \times S/_{\alpha_E \times \alpha_S}}\right) = h^{1,1}\left(\widetilde{E \times S'/_{\alpha_E \times \alpha_{S'}}}\right).$$

Y. Goto, R. Livné, N. Yui in [GLY13] computed zeta function of the Borcea-Voisin threefold:

Theorem 5.1.3 ([GLY13])**.** *The Hasse-Weil zeta function of* $\widetilde{S \times E/_{\alpha_E \times \alpha_S}}$ *has the following form*

$$\xi\left(\widetilde{S \times E/_{\alpha_E \times \alpha_S}}, s\right) = \frac{L(E \otimes \chi, s)L(E \otimes \rho, s)L(J(C_g), s-1)^4}{\xi(\mathbb{Q}, s)L_2(X, s)L_2(X, s-1)\xi(\mathbb{Q}, s-3)}.$$

Moreover if all algebraic cycles in $NS(S)^{\alpha_S}$ *are defined over* $\mathbb{Q}$, *then*

$$L_2(X, s) = \xi(\mathbb{Q}, s-1)^{11+5N-N'},$$

otherwise

$$L_2(X, s) = \xi(\mathbb{Q}, s-1)^{1+t+4(N+1)}L(\rho', s),$$

where $t < 11 + 5N - N' = h^{1,1}$ *is the number of algebraic cycles in* $NS(S)^{\alpha_S}$ *defined over* $\mathbb{Q}$, ρ' *is an irreducible representation of dimension* $11 + 5N - N' - t$ *and* $L(\rho', s)$ *is its Artin L-function.*

Here (S, α_S) is $K3$ surfaces from the family satisfying 2.6.4.

§ 5.2 Generalized Borcea-Voisin construction

In [CG16] A. Cattaneo and A. Garbagnati generalized the Borcea-Voisin construction allowing a non-symplectic automorphisms of a $K3$ surfaces of higher degrees i.e. 3, 4 and 6.

Theorem 5.2.1 ([CG16])**.** *Let* S_d *be a* $K3$ *surface admitting a purely non-symplectic automorphism* α_S *of order* $d = 3, 4, 6$. *Let* E_d *be an elliptic curve admitting an automorphism* α_{E_d} *of order* d. *Then* $S_d \times E_d/_{\alpha_{S_d} \times \alpha_{E_d}^{n-1}}$, *is a singular variety which admits a crepant resolution of singularities* $\widetilde{S_d \times E_d/_{\alpha_{S_d} \times \alpha_{E_d}^{d-1}}}$. *In particular* $\widetilde{S_d \times E_d/_{\alpha_{S_d} \times \alpha_{E_d}^{d-1}}}$ *is a Calabi-Yau threefold.*

Proof. Since S_d and E_d are Calabi-Yau varieties, the product $S_d \times E_d$ has trivial canonical bundle and a generator of $H^{3,0}(S \times E, \mathbb{C})$ is $\omega_{S_d} \wedge \omega_{E_d}$. We have

$$\alpha_{S_d} \times \alpha_{E_d}^{n-1}(\omega_{S_d} \wedge \omega_{E_d}) = \zeta_d \cdot \zeta_d^{d-1}(\omega_{S_d} \wedge \omega_{E_d}) = \omega_{S_d} \wedge \omega_{E_d},$$

hence by 2.1.21 there exists a crepant resolution of ${S_d \times E_d}/{\alpha_{S_d} \times \alpha_{E_d}^{d-1}}$. □

Definition 5.2.2. A crepant resolution of ${S_d \times E_d}/{\alpha_{S_d} \times \alpha_{E_d}^{d-1}} \simeq {S_d \times E_d}/{\mathbb{Z}_d}$ is called a *Calabi-Yau 3-fold of Borcea-Voisin type.*

The authors gave a detailed crepant resolution and computed the Hodge numbers of the resulting algebraic varieties. For all possible orders they computed the Hodge numbers of these varieties and constructed elliptic fibrations on them. Their computations are quite technical and rely on a detailed study of crepant resolutions of threefolds. Main results of their paper are the following theorems:

Order 3:

Theorem 5.2.3 ([CG16]). *Suppose that* $\mathrm{Fix}(\alpha_{S_3})$ *consists of k curves together with a curve with highest genus $g(C)$ and n isolated points, then for any crepant resolution of the variety* ${S_3 \times E_3}/{\alpha_{S_3} \times \alpha_{E_3}^2}$ *the following holds:*

$$h^{1,1} = r + 1 + 3n + 6k \quad \text{and} \quad h^{2,1} = m - 1 + 6g(C).$$

Order 4:

Theorem 5.2.4 ([CG16]). *Suppose* $\mathrm{Fix}_{\alpha_{S_4}^2}$ *is not a union of two elliptic curves, then for any crepant resolution of variety* ${S_4 \times E_4}/{\alpha_{S_4} \times \alpha_{E_4}^3}$ *the following formulas hold*

- *If D is of the first type, then*

$$h^{1,1} = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a, \quad \text{and} \quad h^{1,2} = m - 1 + 7g(D).$$

- *If D is of the second type, then*

$$h^{1,1} = 1 + r + 7k + 3b + 2(n_1 + n_2) + 4a, \quad \text{and} \quad h^{1,2} = m + 2g(D) - \frac{n_2}{2},$$

where

$r :=$ $\dim H^2(S,\mathbb{C})^{\alpha_S}$,

$m :=$ $\dim H^2(S,\mathbb{C})_{\zeta_6^i}$ *for* $1 \leq i \leq 5$,

$N :=$ *number of curves which are fixed by* α_S^2,

$k :=$ *number of curves which are fixed by* α_S *(curves of the first type),*

$b :=$ *number of curves which are fixed by* α_S^2 *and are invariant by* α_S *(curves of the second type),*

$a :=$ *number of pairs* (A, A') *of curves which are fixed by* α_S^2 *and* $\alpha_S(A) = A'$ *(curves of the third type),*

$D :=$ *the curve of the highest genus in* $S^{\alpha_S^2}$,

$n_1 :=$ *number of points which are fixed by* α_S *not laying on the curve* D,

$n_2 :=$ *number of points which are fixed by* α_S *laying on the curve* D.

Order 6:

Theorem 5.2.5 ([CG16]). *For any crepant resolution of variety* $S_6 \times E_6 \big/ \alpha_{S_6} \times \alpha_{E_6}^5$ *the following formulas hold*

$$h^{1,1} = r + 1 + 2l + 2N - 2b + 4k - 2a + 3n' + 3p_{(2,5)} + p_{(3,4)},$$

$$h^{1,2} = \begin{cases} m - 1 + 8g(D) + g(F_2) + g(F_2/\gamma_S) & \text{if } g(D) \geq 1 \\ m - 1 + 2g(G) + 2g(G/\gamma_S) + g(F_1) + g(F_1/\gamma_S) & \\ +g(F_2) + g(F_2/\gamma_S) & \text{if } g(D) = 0 \end{cases}$$

where

$r :=$ $\dim H^2(S,\mathbb{C})^{\gamma_S}$,

$m :=$ $\dim H^2(S,\mathbb{C})_{\zeta_6^i}$ *for* $1 \leq i \leq 5$,

$l :=$ *number of curves fixed by* γ_S,

$k :=$ *number of curves fixed by* γ_S^2,

$N :=$ *number of curves fixed by* γ_S^3,

$p_{(2,5)} + p_{(3,4)} :=$ *number of isolated points fixed by* γ_S *of type* $(2,5)$ *and* $(3,4)$ *i.e. the action of* γ_S *near the point linearises to respectively* $\operatorname{diag}(\zeta_6^2, \zeta_6^5)$ *and* $\operatorname{diag}(\zeta_6^3, \zeta_6^4)$,

$n :=$ *number of isolated points fixed by* γ_S^2,

$2n' :=$ *number of isolated points fixed by* γ_S^2 *and switched by* γ_S,

$F_1, F_2 :=$ *the curves with the highest genus in the fixed locus of* γ_S^3.

$a :=$ *number of triples* (A, A', A'') *of curves fixed by* γ_S^3 *such that* $\gamma_S(A) = A'$ *and* $\gamma_S(A') = A''$,

$b :=$ *number of pairs* (B, B') *of curves fixed by* γ_S^2 *such that* $\gamma_S(B) = B'$,

$D :=$ *the curve with the highest genus in the fixed locus of* γ_S,

$G :=$ *the curve with the highest genus in the fixed locus of* γ_S^2,

In [Bur18] we gave shorter proofs of formulas of Hodge numbers using orbifold cohomology 2.2.1 and orbifold Euler characteristic 2.4.1. We also got new relation among above invariants by comparing Euler characteristic with a stringy Euler characteristic (see section 2.4).

§ 5.3 Higher dimensional Calabi-Yau varieties of Borcea-Voisin type

Take $d \in \{2,3,4,6\}$ and let S_d be a $K3$-surface with non-symplectic automorphism γ_d of order d. Moreover, let E_d be an elliptic curves admitting automorphisms α_d of order d. By [Sil09] no other value of d can be attend by order of an automorphisms of elliptic curve.

The following group

$$G_{d,n} := \{(m_1, m_2, \dots, m_n) \in \mathbb{Z}_d^n : m_1 + m_2 + \dots + m_n = 0\} \simeq \mathbb{Z}_d^{n-1}$$

acts on $S_d \times E_d^{n-1}$ by $\left(\gamma_d\right)^{m_1}$ on the first factor and $\alpha_d^{m_i}$ on the i-th factor, where $2 \leq i \leq n$. Moreover $G_{d,n}$ preserves canonical bundle of $S_d \times E_d^{n-1}$.

Theorem 5.3.1. *Under the above assumptions there exists crepant resolution $Y_{d,n}$ of the quotient variety ${S_d \times E_d^{n-1}}\big/{G_{d,n}}$. In particular $Y_{d,n}$ is $(n+1)$-dimensional Calabi-Yau variety.*

Proof. For $d = 2$, theorem follows from Proposition 2.1 of [CH07] and induction.

Taking $X_1 := E_d$ and $X_2 := S_d$ in Propositions 3.1, 4.1 of [CH07] for $d = 3$ and 4, respectively and 3.2.1 for $d = 6$ we see that all needed assumptions are satisfied. Therefore there exists a crepant resolution $\widetilde{{S_d \times E_d}\big/{\mathbb{Z}_d}}$ of ${S_d \times E_d}\big/{\mathbb{Z}_d}$ and non-symplectic automorphism η_2 on $\widetilde{{S_d \times E_d}\big/{\mathbb{Z}_d}}$ of order d which has the same properties as γ_d. Hence by induction we are done. □

Definition 5.3.2. We call the resulting variety $Y_{d,n}$ an $(n+1)$-*dimensional Calabi-Yau manifold of Borcea-Voisin type.*

Remark 5.3.3. For $n = 2$ our construction coincides with generalised Calabi-Yau threefolds of Borcea-Voisin type given by [CG16]. A. Molnar in his PhD ([Mol15]) studied the above construction for $n = 3$.

5.3.1 Order 6

We shall keep the following notation introduced in [CG16]:

S_6 – $K3$ surfaces with a non-symplectic automorphism $\gamma_6 : S_6 \to S_6$ of order 6

E_6 – elliptic curve with the Weierstrass equation $y^2 = x^3 + 1$, and automorphism $\alpha_6(x, y) = (\zeta_6^2 x, -y)$, where ζ_6 denotes a fixed 6-th root of unity satisfying $\zeta_6^2 = \zeta_3$,

$r = \dim H^2(S_6, \mathbb{C})^{\gamma_6}$,

$m = \dim H^2(S_6, \mathbb{C})_{\zeta_6^i}$ for $i \in \{1, 5\}$,

$\alpha = \dim H^2(S_6, \mathbb{C})_{\zeta_6^i}$ for $i \in \{2, 4\}$,

$\beta = \dim H^2(S_6, \mathbb{C})_{\zeta_6^3}$,

$\mathrm{Fix}(\alpha_6) = \mathrm{Fix}(\alpha_6^3) = \{f_1\}$, $\mathrm{Fix}(\alpha_6^2) = \{f_1, f_2, f_3\}$, where $\alpha_6(f_2) = f_3$, $\alpha_6(f_3) = f_2$ and $\mathrm{Fix}(\alpha_6^3) = \{f_1, f_4, f_5, f_6\}$, where $\alpha_6(f_4) = f_5$, $\alpha_6(f_5) = f_6$ and $\alpha_6(f_6) = f_4$,

$\mathrm{Fix}\left(\gamma_6\right) = \{K_1, \dots K_{\ell-1}\} \cup \{D\} \cup \{P_1, \dots, P_{p_{(2,5)}}\} \cup \{Q_1, \dots, Q_{p_{(3,4)}}\}$, where

- the set $\{K_1, \dots K_{\ell-1}\} \cup \{D\}$ consists of curves which are fixed by γ_6 together with the curve D of maximal genus $g(D)$, in fact K_i are rational,
- $\{P_1, \dots, P_{p_{(2,5)}}\}$ is the set of points such that linearisation of γ_6 near the fixed point is represented by the diagonal matrix $\mathrm{diag}(\zeta_6^2, \zeta_6^5)$,
- $\{Q_1, \dots, Q_{p_{(3,4)}}\}$ is the set of points such that linearisation of γ_6 near the fixed point is represented by the diagonal matrix $\mathrm{diag}(\zeta_6^3, \zeta_6^4)$,

$\mathrm{Fix}\left(\gamma_6^2\right) = \{L_1, \dots, L_{k-2b-1}\} \cup \{G\} \cup \{(A_1, A_1'), \dots, (A_b, A_b')\} \cup \{(R_1, R_1'), \dots, (R_{n'}, R_{n'}')\} \cup \{P_1, \dots, P_{p_{(2,5)}}\}$, where

- the set $\{L_1, \dots L_{k-2b-1}\} \cup \{G\}$ consists of curves which are fixed by γ_6^2 together with the curve G of maximal genus $g(G)$, in fact L_i are rational,
- $\{(A_1, A_1'), \dots, (A_b, A_b')\}$ is the set of all pairs (A_i, A_i') of curves which are fixed by γ_6^2 and $\gamma_6(A_i) = A_i'$ (curves of the third type),
- $\{(R_1, R_1'), \dots, (R_{n'}, R_{n'}')\}$ is the set of pairs of points fixed by γ_6^2 and such that $\gamma_6(R_i) = R_i'$,

$\mathrm{Fix}\left(\gamma_6^3\right) = \{(M_1, M_1', M_1''), \dots, (M_a, M_a', M_a'')\} \cup \{T_1, \dots, T_{N-3a-2}\} \cup \{F_1, F_2\}$, where

- the set $\{(M_1, M_1', M_1''), \dots, (M_a, M_a', M_a'')\}$ consists of curves which are fixed by γ_6^3 and such that $\gamma_6(M_i) = M_i'$, $\gamma_6(M_i') = M_i''$ and $\gamma_6(M_i'') = M_i$,
- the set $\{T_1, \dots, T_{N-3a-2}\} \cup \{F_1, F_2\}$ consists of curves fixed by γ_6 (and so γ_6^3), together with curves F_1 and F_2 of maximal genera $g(F_1)$ and $g(F_2)$, respectively.

$$
\begin{array}{ccc}
H^{i,j}(S_6,\mathbb{C})^{\gamma_6} & H^{i,j}(S_6,\mathbb{C})_{\zeta_6} & H^{i,j}(S_6,\mathbb{C})_{\zeta_6^2} \\
\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & r & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}
&
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & m-1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}
&
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & \alpha & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}
\end{array}
$$

According to the method presented in 2.3 we need to find $F_{S_6,k,j}(X,Y)$ for $0 \leq k, j5$. Let us start with a description of the decomposition of the Hodge diamonds by the eigenspaces of the action of γ_6^* on $H^*(S_6,\mathbb{C})$. Hodge diamonds of eigenspaces have the following shape:

$$
\begin{array}{ccc}
H^{i,j}(S_6,\mathbb{C})_{\zeta_6^3} & H^{i,j}(S_6,\mathbb{C})_{\zeta_6^4} & H^{i,j}(S_6,\mathbb{C})_{\zeta_6^5} \\
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & \beta & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}
&
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & \alpha & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}
&
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & m-1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}
\end{array}
$$

Table 5.1: Hodge diamonds of eigenspaces the action of γ_6^*

Therefore

$$
v_{S_6,0} = \big((XY)^2 + r \cdot XY + 1,\ X^2 + (m-1)\cdot XY,\ \alpha \cdot XY,\ \beta \cdot XY,\ \alpha \cdot XY,\ Y^2 + (m-1)\cdot XY\big).
$$

Fixed locus $\mathrm{Fix}(\gamma_6)$ consists of $\ell - 1$ rational curves, one curve of maximal genus $g(D)$ and $p_{(2,5)} + p_{(3,4)}$ isolated points. The description of that loci in terms of Hodge diamonds is the following:

$$
(\ell - 1) \quad \times \quad \underbrace{\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}}_{\text{rational curve}} \quad + \quad \underbrace{\begin{array}{ccc} & 1 & \\ g(D) & & g(D) \\ & 1 & \end{array}}_{\text{genus } g(D) \text{ curve } D} \quad + \quad \underbrace{p_{(3,4)} + p_{(2,5)}}_{\text{isolated points}}
$$

Locally the action of γ_6 on S_6 along a fixed curve can be diagonalised to a matrix

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_6 & 0 \\ 0 & 0 & \zeta_6^5 \end{pmatrix},
$$

with age equal to 1. In the fixed point of type $(2,5)$ and $(3,4)$ we get respectively matrices

$$\begin{pmatrix} \zeta_6^2 & 0 & 0 \\ 0 & \zeta_6^5 & 0 \\ 0 & 0 & \zeta_6^5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta_6^3 & 0 & 0 \\ 0 & \zeta_6^4 & 0 \\ 0 & 0 & \zeta_6^5 \end{pmatrix},$$

with ages 2. Therefore in the case of points, we must multiply Poincaré polynomials of isolated points (i.e. $p_{(3,4)} + p_{(2,5)}$) by XY. Summarizing we get

$$v_{S_6,1} = \left(\ell + p_{(2,5)} \cdot XY + p_{(3,4)} \cdot XY + g(D) \cdot (X+Y) + \ell \cdot XY,\ 0,\ 0,\ 0,\ 0\right).$$

Since $\mathrm{Fix}(\gamma_6) = \mathrm{Fix}(\gamma_6^5)$ and the fixed points have age equal to 1, we see that

$$v_{S_6,5} = \left(\ell + p_{(2,5)} + p_{(3,4)} + g(D) \cdot (X+Y) + \ell \cdot XY,\ 0,\ 0,\ 0,\ 0\right).$$

In order to compute $v_{S_6,2}$ let us start with Hodge theoretical description of a components of fixed loci of γ_6^2 :

$$(k-1) \quad \times \quad \underbrace{\begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix}}_{\text{rational curve}} \quad + \quad \underbrace{\begin{matrix} & 1 & \\ g(G) & & g(G) \\ & 1 & \end{matrix}}_{\text{genus } g(G) \text{ curve } G} \quad + \quad \underbrace{2n' + p_{(2,5)}}_{\text{isolated points}}$$

Moreover b pairs of curves (A_i, A_i') are permuted by γ_6, therefore $k-2b-1$ rational curves are fixed by γ_6, one curve of genus $g(G)$ is fixed by γ_6 and remaining $2b$ curves form 2-cycles. These gives contribution

$$k - b + g\left({}^{G}\!/_{\gamma_6}\right)(X+Y) + (k-b) \cdot XY$$

to the Poincaré polynomial $F_{S_6,2,0}(X,Y)$ and the contribution equal to

$$b + \left(g(G) - g\left({}^{G}\!/_{\gamma_6}\right)\right) \cdot (X+Y) + b \cdot XY$$

to the Poincaré polynomial $F_{S_6,2,3}(X,Y)$.

Note that fixed point of γ_6^2 have age 2 and n' pairs of them are permuted by γ_6. Therefore, points give contribution $(n' + p_{(2,5)}) \cdot XY$ to $F_{S_6,2,0}(X,Y)$ and contribution $n' \cdot (XY)$ to $F_{S_6,2,3}(X,Y)$. Finally

$$\begin{aligned} v_{S_6,2} = &\Big(k - b + n' \cdot XY + p_{(2,5)} \cdot XY + g\left({}^{G}\!/_{\gamma_6}\right)(X+Y) + (k-b) \cdot XY,\ 0,\ 0, \\ &b + n' \cdot XY + \left(g(G) - g\left({}^{G}\!/_{\gamma_6}\right)\right) \cdot (X+Y) + b \cdot XY,\ 0\Big). \end{aligned}$$

Since $\mathrm{Fix}(\gamma_6^2)=\mathrm{Fix}(\gamma_6^4)$ and the fixed points have age equal to 1 we have:

$$\begin{aligned} v_{S_6,4} =&\Big(k-b+n'+p_{(2,5)}+g\left({}^{G}\!/_{\gamma_6}\right)(X+Y)+(k-b)\cdot XY,\ 0,\ 0,\\ &b+n'+\left(g(G)-g\left({}^{G}\!/_{\gamma_6}\right)\right)\cdot(X+Y)+b\cdot XY,\ 0\Big). \end{aligned}$$

The fixed point locus of γ_6^3 consists of $N-2$ fixed rational curves and 2 curves F_1, F_2 with highest genus $g(F_1)$ and $g(F_2)$, respectively. Hodge diamond of that loci has the following description

$$(N-2)\quad\times\quad\underbrace{\begin{matrix} & 1 & \\ 0 & & 0\\ & 1 & \end{matrix}}_{\text{rational curve}}\quad+\quad\underbrace{\begin{matrix} & 1 & \\ g(F_1) & & g(F_1)\\ & 1 & \end{matrix}}_{\text{genus } g(F_1) \text{ curve } F_1}\quad+\quad\underbrace{\begin{matrix} & 1 & \\ g(F_2) & & g(F_2)\\ & 1 & \end{matrix}}_{\text{genus } g(F_2) \text{ curve } F_2}$$

There are a triplets of rational curves which form 3-cycle under γ_6 action, therefore

$$\begin{aligned} v_{S_6,3} =&\Big(N-2a+\left(g\left({}^{F_1}\!/_{\gamma_6}\right)+g\left({}^{F_2}\!/_{\gamma_6}\right)\right)\cdot(X+Y)+(N-2a)\cdot XY,\ 0,\\ &a+\frac{1}{2}\Big(g(F_1)+g(F_2)-g\left({}^{F_1}\!/_{\gamma_6}\right)-g\left({}^{F_2}\!/_{\gamma_6}\right)\Big)(X+Y)+a\cdot XY,\ 0,\\ &a+\frac{1}{2}\Big(g(F_1)+g(F_2)-g\left({}^{F_1}\!/_{\gamma_6}\right)-g\left({}^{F_2}\!/_{\gamma_6}\right)\Big)(X+Y)+a\cdot XY,\ 0\Big). \end{aligned}$$

From 2.3 and tables 5.2, 5.3 get Poincaré polynomial of $Y_{6,n}$:

$$\begin{aligned} &\sum_{j=0}^{5}\left(F_{S_6,0,j}+\sqrt[6]{XY}F_{S_6,1,j}+\sqrt[6]{(XY)^2}F_{S_6,2,j}+\sqrt[6]{(XY)^3}F_{S_6,3,j}+\sqrt[6]{(XY)^4}F_{S_6,4,j}+\sqrt[6]{(XY)^5}F_{S_6,5,j}\right)\times\\ &\times\left(F_{E,0,j}+\sqrt[6]{XY}F_{E_6,1,j}+\sqrt[6]{(XY)^2}F_{E_6,2,j}+\sqrt[6]{(XY)^3}F_{E_6,3,j}+\sqrt[6]{(XY)^4}F_{E_6,4,j}+\sqrt[6]{(XY)^5}F_{E_6,5,j}\right)^{n-1}=\\ &=\Bigg((XY)^2+r\cdot XY+1+\sqrt[6]{XY}\cdot\Big(\ell+p_{(2,5)}\cdot XY+p_{(3,4)}\cdot XY+g(D)\cdot(X+Y)+\ell\cdot XY\Big)+\\ &+\sqrt[6]{(XY)^2}\cdot\Big(k-b+n'\cdot XY+p_{(2,5)}\cdot XY+g\left(G/\phi_6^{S_6}\right)(X+Y)+(k-b)\cdot XY\Big)+\\ &+\sqrt[6]{(XY)^3}\cdot\Big(N-2a+\left(g\left({}^{F_1}\!/_{\gamma_6}\right)+g\left({}^{F_2}\!/_{\gamma_6}\right)\right)\cdot(X+Y)+(N-2a)\cdot XY\Big)+\\ &+\sqrt[6]{(XY)^4}\cdot\Big(k-b+n'+p_{(2,5)}+g\left({}^{G}\!/_{\gamma_6}\right)(X+Y)+(k-b)\cdot XY\Big)+\\ &+\sqrt[6]{(XY)^5}\cdot\Big(\ell+p_{(2,5)}+p_{(3,4)}+g(D)\cdot(X+Y)+\ell\cdot XY\Big)\Bigg)\cdot\\ &\cdot\Big(1+XY+\sqrt[6]{XY}+2\sqrt[6]{(XY)^2}+2\sqrt[6]{(XY)^3}+2\sqrt[6]{(XY)^4}+\sqrt[6]{(XY)^5}\Big)^{n-1}+ \end{aligned}$$

$k \backslash j$	0	1	2	3	4	5
0	$(XY)^2 + r \cdot XY + 1$	$X^2 + (m-1) \cdot XY$	$\alpha \cdot XY$	$\beta \cdot XY$	$\alpha \cdot XY$	$Y^2 + (m-1) \cdot XY$
1	$\ell + p_{(2,5)} \cdot XY + p_{(3,4)} \cdot XY + g(D) \cdot (X+Y) + \ell \cdot XY$	0	0	0	0	0
2	$k - b + n' \cdot XY + p_{(2,5)} \cdot XY + g\left({}^{G}/_{\gamma_6}\right)(X+Y) + (k-b) \cdot XY$	0	0	$b + n' \cdot XY + \left(g(G) - g\left({}^{G}/_{\gamma_6}\right)\right) \cdot (X+Y) + b \cdot XY$	0	0
3	$N - 2a + \left(g\left({}^{F_1}/_{\gamma_6}\right) + g\left({}^{F_2}/_{\gamma_6}\right)\right) \cdot (X+Y) + (N-2a) \cdot XY$	0	$a + \frac{1}{2}\left(g(F_1) + g(F_2) - g\left({}^{F_1}/_{\gamma_6}\right) - g\left({}^{F_2}/_{\gamma_6}\right)\right)(X+Y) + a \cdot XY$	0	$a + \frac{1}{2}\left(g(F_1) + g(F_2) - g\left({}^{F_1}/_{\gamma_6}\right) - g\left({}^{F_2}/_{\gamma_6}\right)\right)(X+Y) + a \cdot XY$	0
4	$k - b + n' + p_{(2,5)} + g\left({}^{G}/_{\gamma_6}\right)(X+Y) + (k-b) \cdot XY$	0	0	$b + n' + \left(g(G) - g\left({}^{G}/_{\gamma_6}\right)\right) \cdot (X+Y) + b \cdot XY$	0	0
5	$\ell + p_{(2,5)} + p_{(3,4)} + g(D) \cdot (X+Y) + \ell \cdot XY$	0	0	0	0	0

Table 5.2: $F_{S_6,k,j}(X,Y)$

$k \backslash j$	0	1	2	3	4	5
0	$1 + XY$	X	0	0	0	Y
1	1	0	0	0	0	0
2	2	0	0	1	0	0
3	2	0	1	0	1	0
4	2	0	0	1	0	0
5	1	0	0	0	0	0

Table 5.3: $F_{E_6,k,j}(X,Y)$

$$+\Big(X^2+(m-1)\cdot XY\Big)\cdot X^{n-1}+\Bigg(\alpha\cdot XY+\sqrt[6]{(XY)^3}\cdot\Big(a+\frac{1}{2}\Big(g(F_1)+g(F_2)-$$

$$-g\left({F_1}/{\gamma_6}\right)-g\left({F_2}/{\gamma_6}\right)\Big)\cdot(X+Y)+a\cdot XY\Big)\Bigg)\cdot\Big(\sqrt[6]{(XY)^3}\Big)^{n-1}+$$

$$+\Bigg(\beta\cdot XY+\sqrt[6]{(XY)^2}\cdot\Big(b+n'\cdot XY+\Big(g(G)-g\left({G}/{\gamma_6}\right)\Big)\cdot(X+Y)+b\cdot XY\Big)+$$

$$+\sqrt[6]{(XY)^4}\cdot\Big(b+n'+\Big(g(G)-g\left({G}/{\gamma_6}\right)\Big)\cdot(X+Y)+b\cdot XY\Big)\Bigg)\cdot\Big(\sqrt[6]{(XY)^2}+\sqrt[6]{(XY)^4}\Big)^{n-1}+$$

$$+\Bigg(\alpha\cdot XY+\sqrt[6]{(XY)^3}\cdot\Big(a+\frac{1}{2}\Big(g(F_1)+g(F_2)-g\left({F_1}/{\gamma_6}\right)-g\left({F_2}/{\gamma_6}\right)\Big)\cdot(X+Y)+$$

$$+a\cdot XY\Big)\Bigg)\cdot\Big(\sqrt[6]{(XY)^3}\Big)^{n-1}+\Big(Y^2+(m-1)\cdot XY\Big)\cdot Y^{n-1}.$$

From this formula we can compute Euler characteristic by evaluating the above formula at 6-th roots of unity. Therefore we can compute this as a sum of six geometric sequences, hence we expect that there exists recurrence of degree at most 6. In fact two of them coincides and finally we obtain recurrence of order 4.

Corollary 5.3.4. *The Euler characteristic of $Y_{6,n}$ is equal to a_{n-1}, where*

$$\begin{cases} a_n=12a_{n-1}-19a_{n-2}-12a_{n-3}+20a_{n-4},\\ a_0=e(Y_{6,1})=24,\\ a_1=e(Y_{6,2})=4+2r-2m+4l+6p_{(2,5)}+2p_{(3,4)}-4g(D)+8k-4b+6w-\\ \qquad -4g\left({G}/{\gamma_6}\right)-4g(G)+4N-4a-2g\left({F_1}/{\gamma_6}\right)-2g\left({F_2}/{\gamma_6}\right)-2g(F_1)-2g(F_2),\\ a_2=e(Y_{6,3})=80+16r-2m-2\alpha+64l+66p_{(2,5)}+32p_{(3,4)}-64g(D)+68k-64b+36w-\\ \qquad -64g\left({G}/{\gamma_6}\right)-4g(G)+32N-64a-32g\left({F_1}/{\gamma_6}\right)-32g\left({F_2}/{\gamma_6}\right),\\ a_3=e(Y_{6,4})=380+166r-6m-4\alpha+660l+666p_{(2,5)}+330p_{(3,4)}-660g(D)+672k-660b+\\ \qquad +342w-660g\left({G}/{\gamma_6}\right)-12g(G)+332N-660a-330g\left({F_1}/{\gamma_6}\right)-330g\left({F_2}/{\gamma_6}\right)-\\ \qquad -2g(F_1)-2g(F_2). \end{cases}$$

Therefore

$$e(Y_{6,n})=\frac{1}{3}\Bigg(46-r+2m-\alpha+2l+p_{(3,4)}-2g(D)-2k-2b-3w-2g\left({G}/{\gamma_6}\right)+4g(G)-2N-2a-$$

$$-g\left({F_1}/{\gamma_6}\right)-g\left({F_2}/{\gamma_6}\right)+3g(F_1)+3g(F_2)\Bigg)\cdot(-1)^{n-1}-$$

$$-\frac{1}{3}\Bigg(-23+\frac{r}{2}+2m+2\alpha+2l+p_{(3,4)}-2g(D)-2k-2b-3w-2g\left({}^{G}\big/_{\gamma_6}\right)+4g(G)+N-$$
$$-2a-g\left({}^{F_1}\big/_{\gamma_6}\right)-g\left({}^{F_2}\big/_{\gamma_6}\right)\Bigg)\cdot 2^{n-1}+$$
$$+\frac{1}{3}\Bigg(2+r+3\alpha-2l-2p_{(2,5)}-p_{(3,4)}+2g(D)-2k+2b-w+2g\left({}^{G}\big/_{\gamma_6}\right)+2N+2a+$$
$$+g\left({}^{F_1}\big/_{\gamma_6}\right)+g\left({}^{F_2}\big/_{\gamma_6}\right)-3g(F_1)-3g(F_2)\Bigg)-$$
$$-\frac{1}{3}\Bigg(-\frac{r}{2}-2l-2p_{(2,5)}-p_{(3,4)}+2g(D)-2k+2b-w+2g\left({}^{G}\big/_{\gamma_6}\right)-N+2a+$$
$$+g\left({}^{F_1}\big/_{\gamma_6}\right)+g\left({}^{F_2}\big/_{\gamma_6}\right)-1\Bigg).$$

5.3.2 Relations

There are many relations among numerical invariants attached to S_6. Some of them were pointed out in [CG16] and [Bur18]. Most of them follow from adopted notation 5.3.1 and Riemann-Hurwitz formula (see [CG16]). In [Bur18] we got new relation by comparing Stringy Euler Characteristic of $Y_{6,2}$ with the Euler characteristic computed from orbifold Hodge numbers.

In the present section we shall use the same idea as in [Bur18], moreover we shall examine another formulas i.e. Hurwitz formula and holomorphic and topological Lefschetz numbers.

Riemann-Hurwitz formula

Some isolated fixed points of γ_6 lie on curves in Fix $\left(\gamma_6^2\right)$ and all of them are of type $(3,4)$. The canonical map $\pi_G\colon G \to {}^{G}\big/_{\gamma_6}$ can be considered as a covering of G of degree two ramified at

$$p_{(3,4)}-2\cdot\underbrace{(k-2b-\ell-1)}_{\substack{\text{number of rational curves}\\ \text{invariant by }\gamma_6}}$$

points lying on G, which have ramification index 2, therefore by the Reimann-Hurwitz formula we get:

$$2-2g(G)=2\left(2-2g\left({}^{G}\big/_{\phi_6}\right)\right)-\left(p_{(3,4)}-2(k-2b-\ell-1)\right),$$

thus

$$g\left({}^{G}/_{\phi_6}\right) = \frac{1}{4}\left(2g(G) - p_{(3,4)} + 2k - 4b - 2\ell\right). \tag{5.3.1}$$

Assuming $g(D) = 0$. Let $F := F_1 \cup F_2$. The canonical map $\pi_F : F \to {}^{F}/_{\gamma_6}$ is triple covering of F, ramified at

$$p_{(3,4)} + p_{(2,5)} - 2 \cdot \underbrace{(N - 3a - 1 - \ell)}_{\substack{\text{number of rational curves} \\ \text{invariant by } \gamma_6}}$$

points, with ramification index equal to 3, so

$$\begin{aligned} 2 - 2g(F_1) + 2 - 2g(F_2) = 3\left(2 - 2g\left({}^{F_1}/_{\gamma_6}\right) + 2 - 2g\left({}^{F_2}/_{\gamma_6}\right)\right) - \\ - 2(p_{(3,4)} + p_{(2,5)} - 2(N - 3a - 1 - \ell)). \end{aligned}$$

Consequently

(5.3.2)

$$g\left({}^{F_1}/_{\gamma_6}\right) + g\left({}^{F_2}/_{\gamma_6}\right) = \frac{1}{6}\left(2g(F_1) + 2g(F_2) - 2p_{(2,5)} - 2p_{(3,4)} + 4N - 12a - 4l\right).$$

Topological Lefschetz number

Using 2.5 and diagrams 5.1 we deduce:

$$\mathcal{L}_{\text{top}}\left(\gamma_6\right) = 1+1+r+\zeta_6\cdot(1+m-1)+\zeta_6^2\cdot\alpha+\zeta_6^3\cdot\beta+\zeta_6^4\cdot\alpha+\zeta_6^5\cdot(m-1+1) = 2+r+m-\alpha-\beta,$$

therefore

$$2 + r + m - \alpha - \beta = e\left(\text{Fix}\left(\gamma_6\right)\right) = 2\ell - 2g(D) + p_{(3,4)} + p_{(2,5)},$$

which gives relation

$$2 + r + m - \alpha - \beta - 2\ell + 2g(D) - p_{(3,4)} - p_{(2,5)} = 0. \tag{5.3.3}$$

And similarly

$$\begin{aligned} \mathcal{L}_{\text{top}}\left(\gamma_6^2\right) &= (1 + r + \beta + 1) + \zeta_3(1 + m - 1 + \alpha) + \zeta_3^2(\alpha + m - 1 + 1) = \\ &= (1 + r + \beta + 1) - 1 - m - \alpha + 1 = -\alpha + \beta + r + 2 - m, \end{aligned}$$

so

$$-\alpha + \beta + r + 2 - m = e\left(\text{Fix}\left(\gamma_6^2\right)\right) = 2k - 2g(G),$$

giving relation

(5.3.4) $$-\alpha+\beta+r+2-m-2k+2g(G)=0.$$

In the same way:

$$\mathcal{L}_{\text{top}}\left(\gamma_6^3\right)=(1+r+2\alpha+1)-(1+\beta+2m-2+1)=2+r+2\alpha-\beta-2m,$$

so

$$2+r+2\alpha-\beta-2m=e\left(\operatorname{Fix}\left(\gamma_6^3\right)\right)=2N-2g(F_1)-2g(F_2),$$

giving relation

(5.3.5) $$2+r+2\alpha-\beta-2m-2N+2g(F_1)+2g(F_2)=0.$$

Holomorphic Lefschetz number

According to the section 2.5 we shall deduce another relations using holomorphic Lefschetz formulas.

On the one hand

$$\mathcal{L}_{\text{hol}}\left(\gamma_6\right)=\sum_{i=0}^{2}(-1)^i\operatorname{tr}\left(\left(\gamma_6\right)^*|H^i(S,\mathcal{O}_S)\right)=1+\zeta_6^5.$$

Let as compute numbers $a(P)$ and $b(C)$ from theorem 2.5.3:

$$\begin{aligned}
a(P_j)&=\frac{1}{\det\left(1-\gamma_6^*|_{\mathcal{T}_{P_j}}\right)}=\frac{1}{\det\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}\zeta_6^3&0\\0&\zeta_6^4\end{pmatrix}\right)}=\frac{1}{\left(1-\zeta_6^3\right)\left(1-\zeta_6^4\right)},\\
a(Q_j)&=\frac{1}{\det\left(1-\left(\phi_6^{S_6}\right)^*|_{\mathcal{T}_{Q_j}}\right)}=\frac{1}{\det\left(\begin{pmatrix}1&0\\0&1\end{pmatrix}-\begin{pmatrix}\zeta_6^2&0\\0&\zeta_6^5\end{pmatrix}\right)}=\frac{1}{\left(1-\zeta_6^2\right)\left(1-\zeta_6^5\right)}
\end{aligned}$$

and similarly

$$\begin{aligned}
b(K_i)&=\frac{1-g(K_i)}{1-\zeta_6}-\frac{\zeta_6\cdot K_i^2}{\left(1-\zeta_6\right)^2}=\frac{1}{1-\zeta_6}-\frac{\zeta_6\cdot(2g(K_i)-2)}{\left(1-\zeta_6\right)^2}=\frac{1}{1-\zeta_6}-\frac{-2\cdot\zeta_6}{\left(1-\zeta_6\right)^2}=\frac{1+\zeta_6}{\left(1-\zeta_6\right)^2},\\
b(D)&=\frac{1-g(D)}{1-\zeta_6}-\frac{\zeta_6\cdot D^2}{\left(1-\zeta_6\right)^2}=\frac{1-g(D)}{1-\zeta_6}-\frac{\zeta_6\cdot(2-2g(D))}{\left(1-\zeta_6\right)^2}=\frac{(1+\zeta_6)\cdot(1-g(D))}{\left(1-\zeta_6\right)^2}.
\end{aligned}$$

Therefore

$$\mathcal{L}_h\left(\gamma_6\right)=\sum_{i=1}^{\ell-1} b(K_i)+b(D)+\sum_{j=1}^{p_{(2,5)}} a(P_j)+\sum_{j=1}^{p_{(3,4)}} a(Q_j)=$$
$$=(\ell-1)\cdot\frac{1+\zeta_6}{\left(1-\zeta_6\right)^2}+\frac{(1+\zeta_6)\cdot(1-g(D))}{\left(1-\zeta_6\right)^2}+\frac{p_{(3,4)}}{\left(1-\zeta_6^3\right)\left(1-\zeta_6^4\right)}+\frac{p_{(2,5)}}{\left(1-\zeta_6^2\right)\left(1-\zeta_6^5\right)}$$

which is equivalent to

$$1+\zeta_6^5=(\ell-1)\cdot\frac{1+\zeta_6}{\left(1-\zeta_6\right)^2}+\frac{(1+\zeta_6)\cdot(1-g(D))}{\left(1-\zeta_6\right)^2}+\frac{p_{(3,4)}}{\left(1-\zeta_6^3\right)\left(1-\zeta_6^4\right)}+\frac{p_{(2,5)}}{\left(1-\zeta_6^2\right)\left(1-\zeta_6^5\right)}.$$

Multiplying this equality by denominators and after some manipulations we get the following relation:

$$3+3\ell-3g(D)-\frac{p_{(3,4)}}{2}-p_{(2,5)}=0, \tag{5.3.6}$$

which agrees with [Dil12b].

Stringy Euler Characteristic

Using stringy Euler characteristic (see section 2.4) for $n=1,\dots,6$ we have

$$e_s(Y_{6,n})=\begin{cases} 24 & \text{for } n=1,\\ 8l-8g(D)+8p_{(2,5)}+4p_{(3,4)}+8k-8g(G)+8n'+4N-4g(F_1)-4g(F_2) & \text{for } n=2,\\ 96+128l-128g(D)+88p_{(2,5)}+64p_{(3,4)}+48k-48g(G)+ & \\ \quad +48n'+16N-16g(F_1)-16g(F_2) & \text{for } n=3,\\ 672+1320l-1320g(D)+888p_{(2,5)}+660p_{(3,4)}+456k-456g(G)+456n'+ & \\ \quad +168N-168g(F_1)-168g(F_2) & \text{for } n=4,\\ 6720+13312l-13312g(D)+8888p_{(2,5)}+6656p_{(3,4)}+4464k+ & \\ \quad +4464n'-4464g(G)+1664N-1664g(F_1)-1664g(F_2) & \text{for } n=5,\\ 66720+133288l-133288g(D)+88888p_{(2,5)}+66644p_{(3,4)}+44488k+ & \\ \quad +44488n'-44488g(G)+16664N-16664g(F_1)-16664g(F_1) & \text{for } n=6.\end{cases}$$

Comparing $e_s(Y_{6,n})$ and 5.3.4 for $n=1\dots6$ we obtain new relations:

$$-2\alpha+10+N-r-g(F_1)-g(F_2)=0 \tag{5.3.7}$$

$$\begin{aligned}&-m+2+r-2l-p_{(2,5)}-p_{(3,4)}+2g(D)-2b-w-2g\left({}^{G}/_{\gamma_6}\right)+2g(G)\\ &-2a-g\left({}^{F_1}/_{\gamma_6}\right)-g\left({}^{F_2}/_{\gamma_6}\right)+g(F_1)+g(F_2)=0.\end{aligned} \tag{5.3.8}$$

$$\begin{aligned}&-n'-3+\frac{3}{2}\cdot r-6l-2p_{(2,5)}-3p_{(3,4)}+6g(D)+2k-6b-6g\left({}^{G}/_{\gamma_6}\right)+\\ &+4g(G)+\frac{3}{2}\cdot N-6a-3g\left({}^{F_1}/_{\gamma_6}\right)-3g\left({}^{F_2}/_{\gamma_6}\right)+\frac{3}{2}\cdot g(F_1)+\frac{3}{2}\cdot g(F_2)=0.\end{aligned} \tag{5.3.9}$$

To summarize we obtained the following relations:

Proposition 5.3.5. *The following relation holds among parameters attached to* S_6 :

1) $0 = 2m + r + \alpha + \beta - 20,$

2) $0 = n - p_{(2,5)} - 2n',$

3) $0 = 2 + r + m - \alpha - \beta - 2\ell + 2g(D) - p_{(2,5)} - p_{(3,4)},$

4) $0 = -\alpha + \beta + r + 2 - m - 2k + 2g(G),$

5) $0 = 2 + r + 2\alpha - \beta - 2m - 2N + 2g(F_1) + 2g(F_2),$

6) $0 = -2\alpha + 10 + N - r - g(F_1) - g(F_2),$

7) $0 = 3 + 3\ell - 3g(D) - \frac{p_{(3,4)}}{2} - p_{(2,5)},$

8) $0 = -g\left(G/\phi_6^{S_6}\right) + \frac{1}{4}\left(2g(G) - p_{(3,4)} + 2k - 4b - 2\ell\right),$

9) $0 = -g\left(F_1/\phi_6^{S_6}\right) - g\left(F_2/\phi_6^{S_6}\right) + \frac{1}{6}\big(2g(F_1) + 2g(F_2) - 2p_{(2,5)} - 2p_{(3,4)} +$
$+ 4N - 12a - 4l\big),$ (assuming $g(D) = 0$),

10) $0 = -m + 2 + r - 2l - p_{(2,5)} - p_{(3,4)} + 2g(D) - 2b - w - 2g\left({}^{G}/_{\gamma_6}\right) + 2g(G)$
$- 2a - g\left({}^{F_1}/_{\gamma_6}\right) - g\left({}^{F_2}/_{\gamma_6}\right) + g(F_1) + g(F_2),$

11) $0 = -n' - 3 + \frac{3}{2}\cdot r - 6l - 2p_{(2,5)} - 3p_{(3,4)} + 6g(D) + 2k - 6b - 6g\left({}^{G}/_{\gamma_6}\right) +$
$+ 4g(G) + \frac{3}{2}\cdot N - 6a - 3g\left({}^{F_1}/_{\gamma_6}\right) - 3g\left({}^{F_2}/_{\gamma_6}\right) + \frac{3}{2}\cdot g(F_1) + \frac{3}{2}\cdot g(F_2).$

5.3.3 Order 4

We shall keep the following notation coherent with [CG16]:

$S_4 - K3$ surfaces with a non-symplectic automorphism $\gamma_4 : S_4 \to S_4$ of order 4

E_4 – elliptic curve with the Weierstrass equation $y^2 = x^3 + x$, and automorphism α_4 is given by $\alpha_4(x, y) = (-x, iy)$,

$r = \dim H^2(S_4, \mathbb{C})^{\gamma_4},$

$m = \dim H^2(S_4, \mathbb{C})_{\zeta_4^i}$ for $i \in \{1, 2\},$

$\alpha = \dim H^2(S_4, \mathbb{C})_{\zeta_4^2},$

$\mathrm{Fix}(\alpha_4) = \mathrm{Fix}(\alpha_4^3) = \{f_1, f_2\}$, $\mathrm{Fix}(\alpha_4^2) = \{f_1, f_2, f_3, f_4\}$, where $\alpha_4(f_3) = f_4$ and $\alpha_4(f_4) = f_3$,

$\mathrm{Fix}\left(\gamma_4^2\right) = L_1 \cup L_2 \cup \ldots \cup L_{N-b-2a-1} \cup \{D\} \cup \{(A_1, A_1'), (A_2, A_2'), \ldots, (A_a, A_a')\} \cup \{B_1, B_2, \ldots, B_b\}$, where

- $\{(A_1, A_1'), (A_2, A_2'), \ldots, (A_a, A_a')\}$ is the set of all pairs (A_i, A_i') of curves which are fixed by γ_4^2 and $\gamma_4(A_i) = A_i'$,

- $\{B_1, B_2, \dots, B_b\}$ is the set of curves which are fixed by γ_4^2 and are non-invariant by γ_4.

$\mathrm{Fix}\left(\gamma_4\right) = \{R_1, R_2, \dots R_{k-1}\} \cup \{G\} \cup \{P_1, P_2, \dots, P_{n_1}\} \cup \{Q_1, Q_2, \dots, Q_{n_2}\}$, where

- the set $\{R_1, R_2, \dots R_{k-1}\} \cup \{G\}$ contains of curves which are fixed by γ_4 together with the curve G of maximal genus $g(G)$, curves R_i are rational,
- $\{P_1, P_2, \dots, P_{n_1}\}$ is the set of points which are fixed by γ_4 not laying on the curve D,
- $\{Q_1, Q_2, \dots, Q_{n_2}\}$ is the set of points which are fixed by γ_4 laying on the curve D.

From the following description of eigenspaces of $H^{*,*}(S_4, \mathbb{C})$:

$$
\begin{array}{cccc}
H^{i,j}(S_4,\mathbb{C})^{\phi_{\gamma_4}} & H^{i,j}(S_4,\mathbb{C})_{\zeta_4} & H^{i,j}(S_4,\mathbb{C})_{\zeta_4^2} & H^{i,j}(S_4,\mathbb{C})_{\zeta_4^3} \\
\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & r & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} &
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & m-1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} &
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 22-r-2m & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} &
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & m-1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}
\end{array}
$$

We have

$$\nu_{S_4,0} = \Big((XY)^2 + r \cdot XY + 1,\ X^2 + (m-1) \cdot XY,\ (22 - r - 2m) \cdot XY,\ Y^2 + (m-1) \cdot XY\Big).$$

In the fixed locus of γ_4 there are $k - 1$ rational curves and one curve G of maximal genus $g(G)$, and points with linearisation $\mathrm{diag}(1, \zeta_4)$ and $\mathrm{diag}(\zeta_4^2, \zeta_4^3)$ (lying on D or not). In the fixed points we get ages 2, therefore

$$\nu_{S_4,1} = \Big(k + (n_1 + n_2) \cdot XY + g(G) \cdot (X + Y) + k \cdot XY,\ 0,\ 0,\ 0\Big).$$

Similarly

$$\nu_{S_4,3} = \Big(k + (n_1 + n_2) + g(G) \cdot (X + Y) + k \cdot XY,\ 0,\ 0,\ 0\Big).$$

The fixed locus of γ_4^2 consists of $N - b - 2a - 1$ rational curves, one curve D with maximal genus $g(D)$, a pairs of curves switched by γ_4 and fixed curves non-invariant by γ_4. Therefore

$$\nu_{S_4,3} = \Big(N - a + g\left({}^{D}\!/_{\gamma_4}\right) \cdot (X+Y) + (N-a) \cdot XY,\ 0,\ a + \Big(g(D) - g\left({}^{D}\!/_{\gamma_4}\right)\Big)(X+Y) + a \cdot XY\Big).$$

The following tables summaries above facts about $F_{S_4,k,j}(X,Y)$ and $F_{E_4,k,j}(X,Y)$:

From 2.3 and tables 5.4, 5.5 we get Poincaré polynomial of $Y_{4,n}$:

$$\Bigg((XY)^2 + r \cdot XY + 1 + \Big(k + (n_1 + n_2) \cdot XY + g(G) \cdot (X+Y) + k \cdot XY \Big) \cdot \sqrt[4]{XY} +$$
$$+ \Big(N - a + g\left({}^{D}/_{\gamma_4} \right) \cdot (X+Y) + (N-a) \cdot XY \Big) \cdot \sqrt[4]{(XY)^2} + \Big(k + n_1 + n_2 +$$
$$+ g(G) \cdot (X+Y) + k \cdot XY \Big) \cdot \sqrt[4]{(XY)^3} \Bigg) \cdot \Big(1 + XY + 2\sqrt[4]{XY} + 3\sqrt[4]{(XY)^3} + 2\sqrt[4]{(XY)^3} \Big)^{n-1} +$$
$$+ \Big(X^2 + (m-1) \cdot XY \Big) \cdot X^{n-1} + \Big((22 - r - 2m) \cdot XY + \Big(a + \Big(g(D) - g\left({}^{D}/_{\gamma_4} \right) \Big) \Big) (X+Y) +$$
$$+ a \cdot XY \Big) \cdot \sqrt[4]{(XY)^2} \Big) \cdot \Big(\sqrt[4]{(XY)^2} \Big)^{n-1} + \Big(Y^2 + (m-1) \cdot XY \Big) \cdot Y^{n-1}.$$

Similarly as before we obtain the following corollary

Corollary 5.3.6. *The Euler characteristic of $Y_{4,n}$ is equal to a_{n-1}, where*

$$\begin{cases} a_n = 9a_{n-1} + a_{n-2} - 9a_{n-3}, \\ a_0 = e(Y_{4,1}) = 24, \\ a_1 = e(Y_{4,2}) = 2r + 8k + 4n_1 + 4n_2 + 6N - 4a + 4 - 14g(D) - 2m, \\ a_2 = e(Y_{4,3}) = 64 + 20r + 80k + 40n_1 + 40n_2 - 120g(D) + 40N - 40a. \end{cases}$$

Therefore

$$e(Y_{4,n}) = \Big(-N + g(D) + m + 12 \Big) \cdot (-1)^{n-1} - \frac{1}{2}\left(-N + a + 3g(D) - 2k - n_1 - n_2 - \frac{r}{2} - 1 \right) \cdot 9^{n-1} +$$
$$+ \frac{1}{2}\left(N + a + g(D) - 2k - 2m - n_1 - n_2 - \frac{r}{2} + 23 \right).$$

$k \backslash j$	0	1	2	3
0	$(XY)^2 + r \cdot XY + 1$	$X^2 + (m-1) \cdot XY$	$(22 - r - 2m) \cdot XY$	$Y^2 + (m-1) \cdot XY$
1	$k + (n_1 + n_2) \cdot XY + g(G) \cdot (X+Y) + k \cdot XY$	0	0	0
2	$N - a + g\left(D/\gamma_4\right) \cdot (X+Y) + (N-a) \cdot XY$	0	$a + \left(g(D) - g\left(D/\gamma_4\right)\right)(X+Y) + a \cdot XY$	0
3	$k + n_1 + n_2 + g(G) \cdot (X+Y) + k \cdot XY$	0	0	0

Table 5.4: $F_{S_4,k,j}(X,Y)$

$k \backslash j$	0	1	2	3
0	$1 + XY$	X	0	Y
1	2	0	0	0
2	3	0	1	0
3	2	0	0	0

Table 5.5: $F_{E_4,k,j}(X,Y)$

5.3.4 **Order** 3

We shall keep the following notation coherent with [CG16]:

S_3 – $K3$ surfaces with a non-symplectic automorphism $\gamma_3 : S_3 \to S_3$ of order 3,

E_3 – elliptic curve with the Weierstrass equation $y^2 = x^3 + 1$, and automorphism α_3 is given by $\alpha_3(x, y) = (\zeta_3 x, y)$,

$r = \dim H^2(S_3, \mathbb{C})^{\gamma_3}$,

$m = \dim H^2(S_3, \mathbb{C})_{\zeta_3}$,

$\mathrm{Fix}\left(\gamma_3\right) = \mathrm{Fix}\left(\gamma_3^2\right) = \{f_1, f_2, f_3\}$,

$\mathrm{Fix}\left(\gamma_3\right) = L_1 \cup L_2 \cup \ldots \cup L_{k-1} \cup C \cup \{P_1, P_2, \ldots, P_h\}$, where

- the set $\{L_1, L_2, \ldots L_{k-1}\} \cup \{C\}$ consists of curves which are fixed by γ_3 together with the curve C of maximal genus $g(C)$, in fact L_i are rational,
- $\{P_1, P_2, \ldots, P_n\}$ is the set of points which are fixed by γ_3.

$$H^{i,j}(S_3, \mathbb{C})^{\phi_{S_3}} \qquad H^{i,j}(S_3, \mathbb{C})_{\zeta_3} \qquad H^{i,j}(S_3, \mathbb{C})_{\zeta_3^2}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & r & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & m-1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & m-1 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

The following tables (5.6, 5.7) contains Poincaré polynomials $F_{S_3,k,j}(X, Y)$ and $F_{E_3,k,j}(X, Y)$, respectively:

$k \backslash j$	0	1	2
0	$(XY)^2 + r \cdot XY + 1$	$X^2 + (m-1) \cdot XY$	$Y^2 + (m-1) \cdot XY$
1	$k + h \cdot XY + g(C) \cdot (X + Y) + k \cdot XY$	0	0
2	$k + h + g(C) \cdot (X + Y) + k \cdot XY$	0	0

Table 5.6: $F_{S_3,k,j}(X, Y)$

$k \backslash j$	0	1	2
0	$1+XY$	X	Y
1	3	0	0
2	3	0	0

Table 5.7: $F_{E_3,k,j}(X,Y)$

From 2.3 and tables 5.6, 5.7 we get Poincaré polynomial of $Y_{3,n}$:

$$\Bigg((XY)^2 + r\cdot XY + 1 + \Big(k + h\cdot XY + g(C)\cdot(X+Y) + k\cdot XY\Big)\cdot\sqrt[3]{XY} + \Big(k + h +$$
$$+ g(C)\cdot(X+Y) + k\cdot XY\Big)\cdot\sqrt[3]{(XY)^2}\Bigg)\cdot\Big(1 + XY + 3\sqrt[3]{XY} + 3\sqrt[3]{(XY)^2}\Big)^{n-1} +$$
$$+ \Big(X^2 + (m-1)\cdot XY\Big)\cdot X^{n-1} + \Big(Y^2 + (m-1)\cdot XY\Big)\cdot Y^{n-1}.$$

Similarly as before we obtain the following corollary

Corollary 5.3.7. *The Euler characteristic of $Y_{3,n}$ is equal to a_{n-1}, where*

$$\begin{cases} a_n = 7a_{n-1} + 8a_{n-2}, \\ a_0 = e(Y_{3,1}) = 24, \\ a_1 = e(Y_{3,2}) = 2r + 6h + 12k + 4 - 12g(C) - 2m. \end{cases}$$

Therefore

$$e(Y_{3,n}) = \frac{1}{9}\Big(188 - 2r - 6h - 12k + 12g(C) + 2m\Big)\cdot(-1)^{n-1} + \frac{1}{9}\Big(28 + 2r + 6h + 12k - 12g(C) - 2m\Big)\cdot 8^{n-1}.$$

5.3.5 Order 2

S_2 – $K3$ surfaces with a non-symplectic automorphism $\gamma_2 : S_2 \to S_2$ involution,

E_2 – arbitrary elliptic curve with involution $\alpha_2(x,y) = (x,-y)$,

$r = \dim H^2(S_2,\mathbb{C})^{\gamma_2}$,

$m = \dim H^2(S_2,\mathbb{C})_{\zeta_2}$,

$\mathrm{Fix}\left(\alpha_2\right) = \{a,b,c,d\}$,

$\mathrm{Fix}\left(\gamma_2\right) = C_1 \cup C_2 \cup \ldots \cup C_N$ where the set $\{C_1, C_2, \ldots, C_N\}$ contains of curves which are fixed by γ_2 with sum of genera equals N'.

The following tables (5.8, 5.9) contains Poincaré polynomials $F_{S_2,k,j}(X,Y)$ and $F_{E_2,k,j}(X,Y)$, respectively:

$k \backslash j$	0	1
0	$(XY)^2 + r \cdot XY + 1$	$X^2 + Y^2 + (m-2) \cdot XY$
1	$N + N' \cdot (X+Y) + N \cdot XY$	0

Table 5.8: $F_{S_2,k,j}(X,Y)$

$k \backslash j$	0	1
0	$1 + XY$	$X + Y$
1	4	0

Table 5.9: $F_{E_2,k,j}(X,Y)$

From 2.3 it follows that Poincaré polynomial of $Y_{2,n}$ equals

$$\left((XY)^2 + r \cdot XY + 1 + \Big(N + N' \cdot (X+Y) + N \cdot XY\Big) \cdot \sqrt{XY}\right) \cdot \Big(1 + XY + 4\sqrt{XY}\Big)^{n-1} + \\ + \Big(X^2 + Y^2 + (m-2) \cdot XY\Big) \cdot (X+Y)^{n-1}.$$

Using the following relations (see [Voi93]):

$$r = 10 + N - N' \quad \text{and} \quad m = 12 - N + N'$$

we can rewrite the above formula in terms of N and N', i.e.

$$\left((XY)^2 + (10 + N - N') \cdot XY + 1 + \Big(N + N' \cdot (X+Y) + N \cdot XY\Big) \cdot \sqrt{XY}\right) \times \\ \times \Big(1 + XY + 4\sqrt{XY}\Big)^{n-1} + \Big(X^2 + Y^2 + (10 - N + N') \cdot XY\Big) \cdot (X+Y)^{n-1}.$$

Remark 5.3.8. From Nikulin's classification ([Nik87]) it follows that for any $K3$ surface S with non-symplectic involution which fixes N curves with sum of genera equal to N', there exists a mirror $K3$ surface S' and its non-symplectic involution which fixes N' curves with sum of genera equal to N. Hodge diamonds of $\widetilde{S \times E/\mathbb{Z}_2}$ and $\widetilde{S' \times E/\mathbb{Z}_2}$ are equal to respectively:

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 11+5N'-N & & 0 & \\
1 & & 11+5N-N' & & 11+5N-N' & & 1 \\
 & 0 & & 11+5N'-N & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 11+5N-N' & & 0 & \\
1 & & 11+5N'-N & & 11+5N'-N & & 1 \\
 & 0 & & 11+5N-N' & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 5.1: Hodge diamonds of Borcea-Voisin mirror pair

Corollary 5.3.9. *The Euler characteristic of $Y_{2,n}$ is equal to a_{n-1}, where*

$$
\begin{cases}
a_n = 4a_{n-1} + 12a_{n-2}, \\
a_0 = e(Y_{3,1}) = 24, \\
a_1 = e(Y_{3,2}) = 12N - 12N'.
\end{cases}
$$

Therefore

$$e(Y_{2,n}) = \frac{1}{2}(12 + 3N - 3N') \cdot 6^{n-1} + \frac{1}{2}(36 - 3N + 3N') \cdot (-2)^{n-1}.$$

§ 5.4 Explicit computation of the zeta function

We shall give details of computation of the Zeta functionof a Borcea-Voisin Calabi-Yau n-fold with a very particular shape of the Hodge diamond.

Let S be the $K3$ surface no. 18 from Table 1 of [Dil12b]. Then S is isomorphic to an elliptic $K3$ surface $X \to \mathbb{P}^1$ whose Weierstrass equation is

$$y^2 = x^3 + \lambda(z-1)^2 z^5$$

and on which we have the following ζ_6-action:

$$\alpha : (x, y, t) \to (\zeta_3^2 x, y, z).$$

The surface S has the following invariants (see [Dil12b] and relations 5.3.5):

r	m	n	n'	k	a	$p_{3,4}$	$p_{2,5}$	ℓ	N	b	α	β
19	1	9	0	6	0	6	9	3	10	0	0	1

The Hodge diamonds of the respective eigenspaces of induced ζ_6-action on $H^{**}(S, \mathbb{C})$ have the following forms:

$$H^{i,j}(S,\mathbb{C})^{\alpha} \qquad\qquad H^{i,j}(S,\mathbb{C})_{\alpha} \qquad\qquad H^{i,j}(S,\mathbb{C})_{\alpha^2}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 19 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$$H^{i,j}(S,\mathbb{C})_{\alpha^3} \qquad\qquad H^{i,j}(S,\mathbb{C})_{\alpha^4} \qquad\qquad H^{i,j}(S,\mathbb{C})_{\alpha^5}$$

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 1 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

The elliptic fibrations $S \to \mathbb{P}^1$ has 1 type IV fiber and 2 type II^* fibers. To study zeta function of S we need careful analysis of the resolution of singularities of

$$y^2 = x^3 + \lambda(z-1)^2 z^5.$$

In fact it suffices to study resolution of singularities of $y^2 = x^3 + \lambda \cdot z^2$ and $y^2 = x^3 + \lambda \cdot z^5$.

5.4.1 Resolution of singularities of $y^2 = x^3 + \lambda \cdot z^2$

Let us describe blow up resolution of singularities of $F_1 := \{y^2 = x^3 + \lambda \cdot z^2\}$. Process of the resolution can be displayed in the following graph:

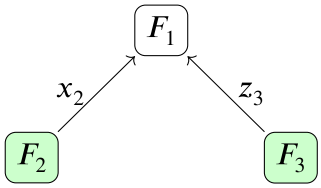
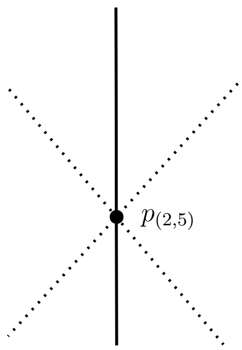


Table 5.10: Process of the resolution of $y^2 = x^3 + \lambda \cdot z^2$

where green boxes denote smooth affine pieces with the following details of a blow-up maps:

$$
\begin{aligned}
F_2 = \{y_2^2 =& \lambda z_2^2 + x_2\} : \\
& \text{reverse map: } (x, y, z) \to (x_2, x_2 y_2, x_2 z_2), \\
& \text{inverse map: } (x_2, y_2, z_2) \to \left(x, \frac{y}{x}, \frac{z}{x}\right), \\
& \zeta_6\text{-action: } \left(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z\right) = \left(\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z\right). \\
F_3 = \{y_3^2 =& x_3^3 z_3 + \lambda\} : \\
& \text{map: } (x, y, z) \to (x_3 z_3, y_3 z_3, z_3), \\
& \text{inverse map: } (x_3, y_3, z_3) \to \left(\frac{x}{z}, \frac{y}{z}, z\right), \\
& \zeta_6\text{-action: } \left(\zeta_3 x, -y, z\right) = \left(\zeta_6^2 x, \zeta_6^3 y, z\right).
\end{aligned}
$$

In any affine part let us mark exceptional fiber in red. Graph of intersections of curves in the resolution has the following shape:



Dotted curves are permutated by α and the last one is fixed.

5.4.2 Resolution of singularities of $y^2 = x^3 + \lambda \cdot z^4$

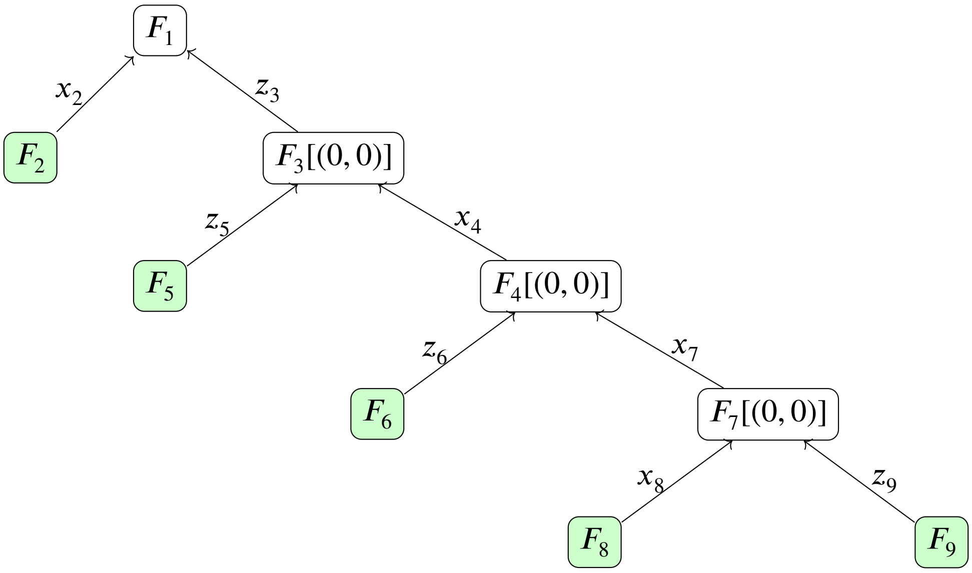


Table 5.11: Process of the resolution of $y^2 = x^3 + \lambda \cdot z^4$

Reverse maps of smooth affine pieces:

$$F_2 = \{y_2^2 = \lambda x_2^2 z_2^4 + x_2\} :$$

$$\text{map: } (x, y, z) \to (x_2, x_2 y_2, x_2 z_2),$$

$$\text{inverse map: } (x_2, y_2, z_2) \to \left(x, \frac{y}{x}, \frac{z}{x}\right),$$

$$\text{action: } \left(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z\right) = \left(\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z\right).$$

$$F_5 = \{y_5^2 = x_5^3 z_5^2 + \lambda\} :$$

$$\text{map: } (x, y, z) \to (x_5 z_5^2, y_5 z_5^2, z_5)$$

$$\text{inverse map: } (x_5, y_5, z_5) \to \left(\frac{x}{z^2}, \frac{y}{z^2}, z\right),$$

$$\text{action: } \left(\zeta_3 x, -y, z\right) = \left(\zeta_6^2 x, \zeta_6^3 y, z\right).$$

$$F_6 = \{y_6^2 = x_6^2 z_3 + \lambda\} :$$

$$\text{map: } (x, y, z) \to (x_6^2 z_6^3, x_6^2 y_6 z_6^4, x_6 z_6^2),$$

$$\text{inverse map: } (x_6, y_6, z_6) \to \left(\frac{x^2}{z^3}, \frac{y}{z^2}, \frac{z^2}{x}\right),$$

$$\text{action: } \left(\zeta_3^2 x, -y, \zeta_3^2 z\right) = \left(\zeta_6^4 x, \zeta_6^3 y, \zeta_6^4 z\right).$$

$$F_8 = \{y_8^2 = \lambda z_8^2 + z_8\} :$$

$$\text{map: } (x, y, z) \to (x_8^4 z_8, x_8^6 y z_8, x_8^3 z_8),$$

inverse map: $(x_8, y_8, z_8) \to \left(\frac{x}{z}, \frac{yz^2}{x^3}, \frac{z^4}{x^3}\right)$,

action: $(\zeta_3 x, -y, z) = (\zeta_6^2 x, \zeta_6^3 y, z)$.

$F_9 = \{y_9^2 = \lambda + x_9\}$:

map: $(x, y, z) \to (x_9^3 z_9^4, x_9^4 y_9 z_9^6, x_9^2 z_9^3)$,

inverse map: $(x_9, y_9, z_9) \to \left(\frac{x^3}{z^4}, \frac{y}{z^2}, \frac{z^3}{x^2}\right)$,

action: $(x, -y, \zeta_3 z) = (x, \zeta_6^3 y, \zeta_6^2 z)$.

5.4.3 Resolution of singularities of $y^2 = x^3 + \lambda \cdot z^5$

In this section we give details of a resolution of singularities of the elliptic surface $y^2 = x^3 + \lambda \cdot z^5$.

The graph 5.12 displays graph of resolution of singularities of fiber. The arrow means blow up, variables x_i (or z_i) denote blow up map realized by respective variable. If there is an affine piece with more than one singularity, we treat this as two separate pieces.

Reverse maps of smooth affine pieces:

$F_2 = \{y_2^2 = \lambda x_2^3 z_2^5 + x_2\}$:

map: $(x, y, z) \to (x_2, x_2 y_2, x_2 z_2)$,

inverse map: $(x_2, y_2, z_2) \to \left(x, \frac{y}{x}, \frac{z}{x}\right)$,

exceptional fiber: $x_2 z_2$,

action: $(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z) = (\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z)$.

$F_5 = \{y_5^2 = x_5^3 z_5^2 + \lambda z_5\}$:

map: $(x, y, z) \to (x_5 z_5^2, y_5 z_5^2, z_5)$,

inverse map: $(x_5, y_5, z_5) \to \left(\frac{x}{z^2}, \frac{y}{z^2}, z\right)$,

action: $(\zeta_3 x, -y, z) = (\zeta_6^2 x, \zeta_6^3 y, z)$.

$F_{10} = \{y_{10}^2 = \lambda x_{10}^3 z_{10}^3 + z_{10}\}$:

map: $(x, y, z) \to (x_{10}^4 z_{10}, x_{10}^6 y_{10} z_{10}, x_{10}^3 z_{10})$,

inverse map: $(x_{10}, y_{10}, z_{10}) \to \left(\frac{x}{z}, \frac{yz^2}{x^3}, \frac{z^4}{x^3}\right)$,

action: $(\zeta_3 x, -y, z) = (\zeta_6^2 x, \zeta_6^3 y, z)$.

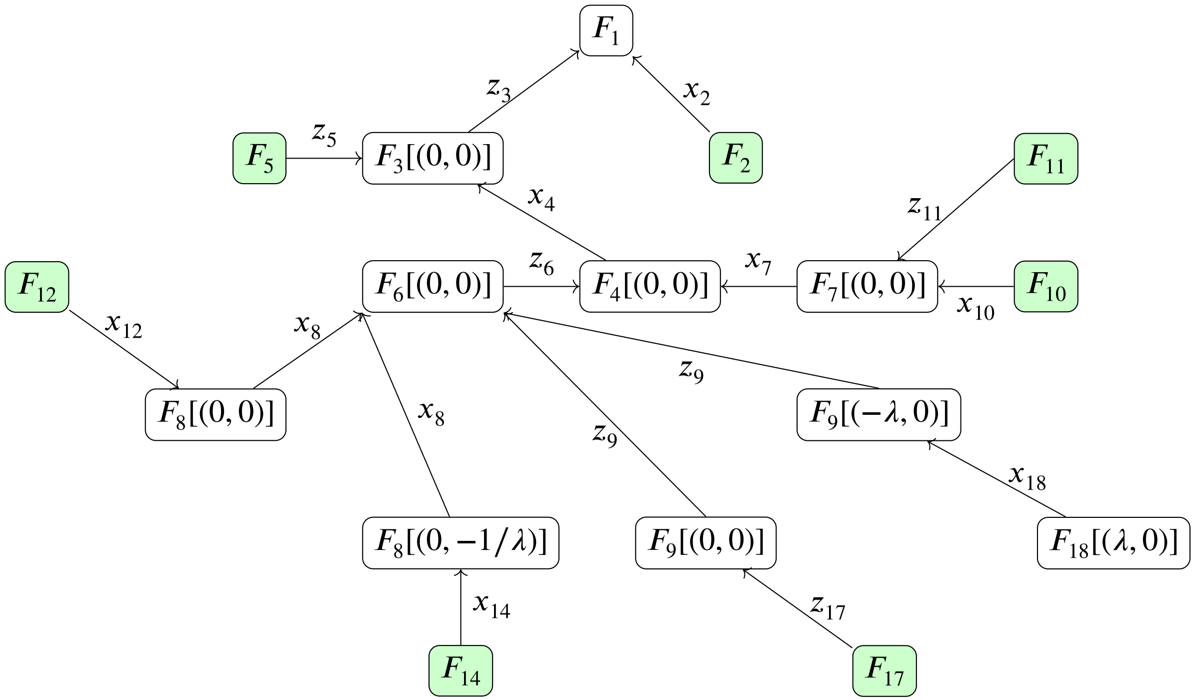


Table 5.12: Process of the resolution of $y^2 = x^3 + \lambda \cdot z^5$

$$F_{11} = \{y_{11}^2 = \lambda x_{11}^2 z_{11}^3 + x_{11}\} :$$

map: $(x, y, z) \to (x_{11}^3 z_{11}^4, x_{11}^4 y_{11} z_{11}^6, x_{11}^2 z_{11}^3)$,

inverse map: $(x_{11}, y_{11}, z_{11}) \to \left(\frac{x^3}{z^4}, \frac{y}{z^2}, \frac{z^3}{x^2} \right)$,

action: $\left(x, -y, \zeta_3 z\right) = \left(x, \zeta_6^3 y, \zeta_6^2 z\right)$.

$$F_{12} = \{y_{12}^2 = \lambda x_{12} z_{12}^2 + z_{12}\} :$$

map: $(x, y, z) \to (x_{12}^8 z_{12}^3, x_{12}^{12} y_{12} z_{12}^4, x_{12}^5 z_{12}^2)$

inverse map: $(x_{12}, y_{12}, z_{12}) \to \left(\frac{x^2}{z^3}, \frac{yz^4}{x^4}, \frac{z^8}{x^5} \right)$,

action: $\left(\zeta_3^2 x, -\zeta_3^2 y, \zeta_3 z\right) = \left(\zeta_6^4 x, \zeta_6 y, \zeta_6^2 z\right)$.

$$F_{14} = \{y_{14}^2 = \lambda x_{14} z_{14}^2 - z_{14}\} :$$

map: $(x, y, z) \to (x_{14}^5 (x_{14} z_{14} - 1/\lambda)^3, x_{14}^8 y_{14} (z_{14} z_{14} - 1/\lambda)^4, x_{14}^3 (x_{14} z_{14} - 1/c)^2)$,

inverse map: $(x_{14}, y_{14}, z_{14}) \to \left(\frac{x^2}{z^3}, \frac{yz^4}{x^4}, \frac{z^3(cz^5 + x^3)}{cx^5} \right)$,

action: $\left(\zeta_3^2 x, -\zeta_3^2 y, \zeta_3 z\right) = \left(\zeta_6^4 x, \zeta_6 y, \zeta_6^2 z\right)$.

$$F_{17} = \{y_{17}^2 = x_{17}^2 z_{17} + \lambda x_{17}\} :$$

map: $(x, y, z) \to (x_{17}^2 z_{17}^7, x_{17}^2 y_{17} z_{17}^{10}, x_{17} z_{17}^4)$,

inverse map: $(x_{17}, y_{17}, z_{17}) \to \left(\frac{x^4}{z^7}, \frac{yx^2}{z^6}, \frac{z^2}{x} \right)$,

action: $\left(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z\right) = \left(\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z\right)$.

$$F_{19} = \{y_{19}^2 = x_{19}^2 z_{19} - \lambda x_{19}\} :$$

map: $(x, y, z) \to \left((x_{19} z_{19}^7 - \lambda)^2 z_{19}^5, (x_{19} z_{19} - \lambda)^2 y_{19} z_{19}^8, (x_{19} z_{19} - \lambda) z_{19}^3\right)$,

inverse map: $(x_{19}, y_{19}, z_{19}) \to \left(\frac{(\lambda \cdot z^5 + x^3)x}{z^7}, \frac{yx^2}{z^6}, \frac{z^2}{x} \right)$,

action: $\left(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z\right) = \left(\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z\right)$.

$$F_{20} = \{y_{20}^2 = \lambda z_{20}\} :$$

map: $(x, y, z) \to \left((x_{20} z_{20} - 1/\lambda)^8 x_{20}^5, x_{20}^8 (x_{20} z_{20} - 1/\lambda)^{12} y_{20}, x_{20}^3 (x_{20} z_{20} - 1/\lambda)^5\right)$,

inverse map: $(x_{20}, y_{20}, z_{20}) \to \left(\frac{x^5}{z^8}, \frac{yz^4}{x^4}, \frac{z^8(\lambda z^5 + x^3)}{\lambda x^8} \right)$,

action: $\left(\zeta_3^2 x, -\zeta_3^2 y, \zeta_3 z\right) = \left(\zeta_6^4 x, \zeta_6 y, \zeta_6^2 z\right)$.

$$F_{21} = \{y_{21}^2 = \lambda x_{21}\} :$$

map: $(x,y,z) \to \left(x_{21}z_{21}^5(z_{21}-1/\lambda)^8, x_{21}^7 z_{21}^8(z_{21}-1/\lambda)^{12}, x_{21}^3 z_{21}^3(z_{21}-1/\lambda)^5\right)$,

inverse map: $(x_{21},y_{21},z_{21}) \to \left(\dfrac{\lambda x^8}{z^8(\lambda z^5+x^3)}, \dfrac{\lambda x^4 y}{w^4(\lambda z^5+x^3)}, \dfrac{\lambda z^5+x^3}{\lambda x^3}\right)$

action: $\left(\zeta_3^2 x, -\zeta_3 y, z\right) = \left(\zeta_6^4 x, \zeta_6^5 y, z\right)$.

$F_{22} = \{y_{22}^2 = \lambda z_{22}\}$:

map: $(x,y,z) \to \left(x_{22}^8(x_{22}z_{22}+1/\lambda)^8 z_{22}^3, x_{22}^{12}(x_{22}z_{22}+1/\lambda)^8 y_{22}z_{22}^4, x_{22}^5(x_{22}z_{22}+1/\lambda)^3 z_{22}^2\right)$,

inverse map: $(x_{22},y_{22},z_{22}) \to \left(\dfrac{\lambda x^5}{z^3(\lambda z^5+x^3)}, \dfrac{yz^4}{x^4}, \dfrac{(\lambda z^5+x^3)z^8}{\lambda x^8}\right)$,

action: $\left(\zeta_3^2 x, -\zeta_3 y, \zeta_3 z\right) = \left(\zeta_6^4 x, \zeta_6^5 y, \zeta_6^2 z\right)$.

$F_{23} = \{y_{23}^2 = \lambda x_{23}\}$:

map: $(x,y,z) \to \left(x_{23}^5 z_{23}^8(z_{23}+1/\lambda)^5, x_{23}^7 z_{23}^{12}(z_{23}+1/\lambda)^8 y_{23}, x_{23}^3 z_{23}^5(z_{23}+1/\lambda)^3\right)$,

inverse map: $(x_{23},y_{23},z_{23}) \to \left(\dfrac{\lambda x^8}{z^8(\lambda z^5+x^3)}, \dfrac{\lambda x^4 y}{z^4(\lambda z^5+x^3)}, \dfrac{z^5}{x^3}\right)$,

action: $\left(\zeta_3^2 x, -\zeta_3 y, z\right) = \left(\zeta_6^4 x, \zeta_6^5 y, z\right)$.

$F_{24} = \{y_{24}^2 = z_{24}\}$:

map: $(x,y,z) \to \left(x_{24}^5 z_{24}^5(x_{24}-\lambda)^7, x_{24}^8 y_{24} z_{24}^7(x_{24}-\lambda)^{10}, x_{24}^3 z_{24}^3(x_{24}-\lambda)^4\right)$,

inverse map: $(x_{24},y_{24},z_{24}) \to \left(\dfrac{x^3+\lambda z^5}{z^5}, \dfrac{yz^6}{x^2(\lambda z^5+x^3)}, \dfrac{z^{12}}{x^4(\lambda z^5+x^3)}\right)$,

action: $\left(x, -\zeta_3 y, \zeta_3^2 z\right) = \left(x, \zeta_6^5 y, \zeta_6^4 z\right)$.

$F_{25} = \{y_{25}^2 = x_{25}\}$:

map: $(x,y,z) \to \left((x_{25}z_{25}-\lambda)^7 z_{25}^5, (x_{25}z_{25}-\lambda)^{10} y_{25} z_{25}^8, (x_{25}z_{25}-\lambda)^4 z_{25}^3\right)$,

inverse map: $(x_{25},y_{25},z_{25}) \to \left(\dfrac{x^4(\lambda z^5+x^3)}{z^{12}}, \dfrac{yx^2}{z^6}, \dfrac{z^7}{x^4}\right)$,

action: $\left(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z\right) = \left(\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z\right)$.

$F_{26} = \{y_{26}^2 = z_{26}\}$:

map: $(x,y,z) \to \left(x_{26}^7(x_{26}+\lambda)^5 z_{26}^5, x_{26}^{10}(x_{26}+\lambda)^8 y_{26} z_{26}^7, x_{26}^4(x_{27}+\lambda)^3 z_{26}^3\right)$,

inverse map: $(x_{26},y_{26},z_{26}) \to \left(\dfrac{x^3}{z^5}, \dfrac{yz^6}{x^2(\lambda z^5+x^3)}, \dfrac{z^{12}}{x^4(\lambda z^5+x^3)}\right)$,

action: $\left(x, -\zeta_3 y, \zeta_3^2 z\right) = \left(x, \zeta_6^5 y, \zeta_6^4 z\right)$.

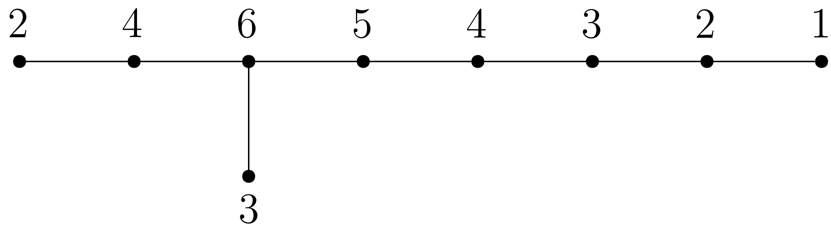
$F_{27} = \{y_{27}^2 = x_{27}\}$:

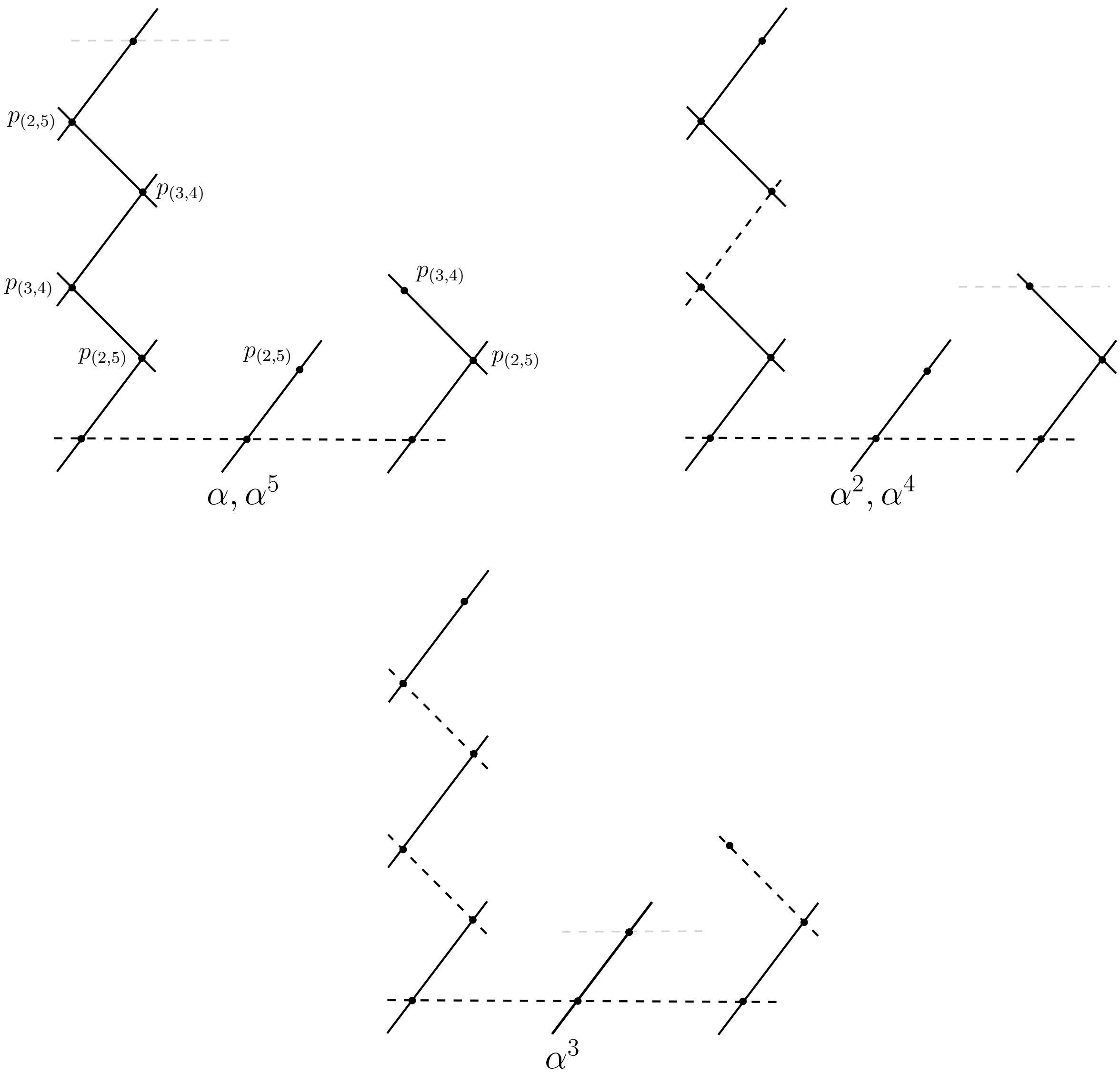
map: $(x,y,z) \to \left(x_{27}^2(x_{27}z_{27}+\lambda)^5 z_{27}^7, x_{27}^2(x_{27}z_{27}+\lambda)^8 y_{27} z_{27}^{10}, x_{27}(x_{27}z_{27}+\lambda)^3 z_{27}^4\right)$,

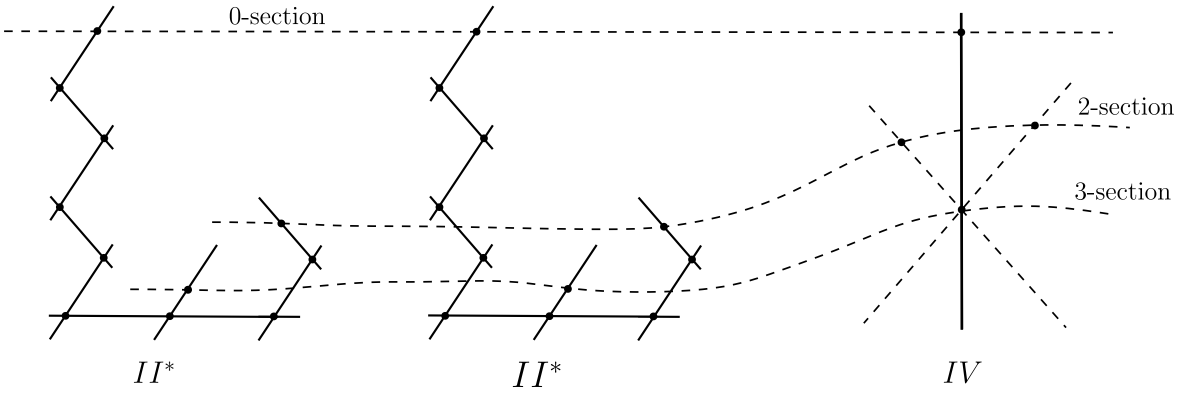
inverse map: $(x_{27},y_{27},z_{27}) \to \left(\dfrac{x^4(\lambda z^5+x^3)}{z^{12}}, \dfrac{yx^2}{z^6}, \dfrac{z^7}{x(\lambda z^5+x^3)}\right)$

action: $\left(\zeta_3 x, -\zeta_3^2 y, \zeta_3^2 z\right) = \left(\zeta_6^2 x, \zeta_6 y, \zeta_6^4 z\right)$.

Intersections of the curves may be presented in the following Dynkin diagram ($\widetilde{E_8}$).







Denote by C_1 the 0-section, $C_2, \ldots, C_{20}$ curves in the fibers (see picture 5.2). Given set of curves is linearly independent since the matrix of intersection is non-singular:

$$
\det\begin{pmatrix}
-2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2
\end{pmatrix} = -3.
$$

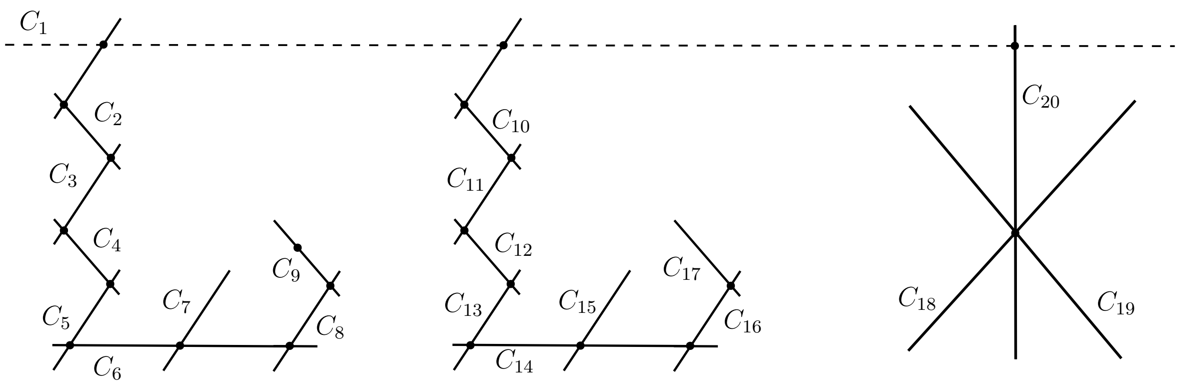


Figure 5.2: Curves in the fibers

§ 5.5 Zeta function for $d = 2$

In the presenting section we compute zeta function of classical Borcea-Voisin Calabi-Yau threefold.

Let (S, α_S) be a $K3$ surface admitting a non-symplectic involution α_S. Consider an elliptic curve E with non-symplectic involution α_E. Let us denote by $H^2(S, \mathbb{C})^{\alpha_S}$ the invariant part of cohomology $H^2(S, \mathbb{C})$ under α_S and by r the dimension $r = \dim H^2(S, \mathbb{C})^{\alpha_S}$. We also denote the eigenspace for -1 of the induced action α_S^* on $H^2(S, \mathbb{C})$ by $H^2(S, \mathbb{C})_{-1}$ and by m the dimension $m = \dim H^2(S, \mathbb{C})_{-1}$.

We use the notation adopted in the section 2.7.

The polynomial $Z_{E,0,0}$: In that case:

$$Z_{E,0,0} = Z_q\left(H^{**}(E)^{\alpha_E}\right) = \frac{1}{(1-T)(1-qT)},$$

since the Hodge diamond of α_E invariant part is equal to

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$

The polynomial $Z_{E,0,1}$: In that case:

$$Z_{E,0,0} = Z_q\left(H^{**}(E)_{-1}\right) = 1 - a_q T + qT^2,$$

for some integer a_q, since the Hodge diamond of α_E invariant part is equal to

$$\begin{array}{ccc} & 0 & \\ 1 & & 1 \\ & 0 & \end{array}$$

The polynomial $Z_{E,0,1}$: In that case:

$$Z_{E,1,0} = Z_q\left(H^{**}\left(\mathrm{Fix}(\alpha_E)\right)\right).$$

The last polynomial depends on the number of fixed points in $\mathrm{Fix}(\alpha_E) := \{a, b, c, d\}$ which are defined over $\mathbb{F}_q$:

- All points a, b, c, d are defined over $\mathbb{F}_q$. Then the Zeta function is equal to

$$(1-T)^4.$$

- Points a, b, $c+d$ are defined over $\mathbb{F}_q$ but c, d are not defined over $\mathbb{F}_q$. The action has the following linearisation

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\sim\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},$$

with characteristic polynomial

$$(1-T)^3(1+T).$$

- a is defined over $\mathbb{F}_q$ and b, c, d not. The action has the following linearisation

$$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}\sim\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\zeta_3&0\\0&0&0&\zeta_3^2\end{pmatrix},$$

with characteristic polynomial

$$(1-T)^2(1+T+T^2).$$

The polynomial $Z_{E,1,1}$: This polynomial is obviously equal to 1.

Therefore table corresponding to elliptic curve E has the following form:

$k \backslash j$	0	1
0	$\frac{1}{(1-T)(1-qT)}$	$1-a_qT+qT^2$
1	$\frac{1}{(1-T)^4}$, $\frac{1}{(1-T)^3(1+T)}$, $\frac{1}{(1-T)^2(1+T+T^2)}$	1

Table 5.13: $Z_{E,k,j}(T)$

Now let us make analysis of corresponding table for $K3$ surface S.

The polynomial $Z_{S,0,0}$: In that case:

$$Z_{S,0,0}=Z_q(H^{**}(S)^{\alpha_S})$$

The Frobenius map acting on r curves induces permutation $\pi \in S_r$ with decomposition into disjoint cycles of lengths $a_1, a_2, \dots, a_s$ for some natural s. Since the Hodge diamond of α_S invariant part is equal to

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & r & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

we have

$$Z_{S,0,0} = \frac{1}{(1-T)(1-qT)\prod_{i=1}^{s}\left(1-(qT)^{a_i}\right)\left(1-q^2T\right)}.$$

The polynomial $Z_{S,0,1}$: One can see that

$$H^2_{\text{ét}}\left(S, \mathbb{Q}_l\right) = T(S) \oplus NS(S)_{-1}.$$

In general we cannot say much more but in special cases we can find quit explicit description of the polynomial $Z_{S,0,1}$. In particular if S is one of $K3$ surfaces appearing in Borcea-Voisin construction then the poly can read out from Theorem 2.6.4. We shall compute polynomial in two particular cases in sections 5.5.1 and 5.4.

The polynomial $Z_{S,1,0}$: Let C_g be the curve of maximal genus g in $\text{Fix}(\alpha_S)$. Then we see that

$$Z_{S,1,0} = Z_q\left(H^*_{\text{ét}}\left(\text{Fix}(\alpha_S)\right)\right) = \frac{\det\left(1 - t\cdot\text{Frob}_q\,|H^1_{\text{ét}}(C_g, \mathbb{Q}_l)\right)}{(1-T)(1-qT)\prod_{i=1}^{s}(1-(qT)^{a_s})}.$$

Otherwise the local zeta function $Z_{S,1,0}$ is equal to

$$\frac{1}{(1-T)(1-qT)\prod_{i=1}^{s}(1-(qT)^{a_s})}.$$

The polynomial $Z_{S,1,1}$: This polynomial is obviously equal to 1.

Therefore the corresponding table 2.2 is equal to

$k \backslash j$	0	1
0	$\dfrac{1}{(1-T)(1-qT)\prod_{i=1}^{s}(1-(qT)^{a_i})\left(1-q^2T\right)}$	$Z_{S,0,1}$
1	$\dfrac{\det\left(1-t\cdot \mathrm{Frob}_q \mid H^1_{\text{ét}}(C_g,\mathbb{Q}_l)\right)}{(1-T)(1-qT)\prod_{i=1}^{s}\left(\left(1-T^{a_i}\right)\left(1-(qT)^{a_i}\right)\right)}$	1

Table 5.14: $Z_{S,k,j}(T)$

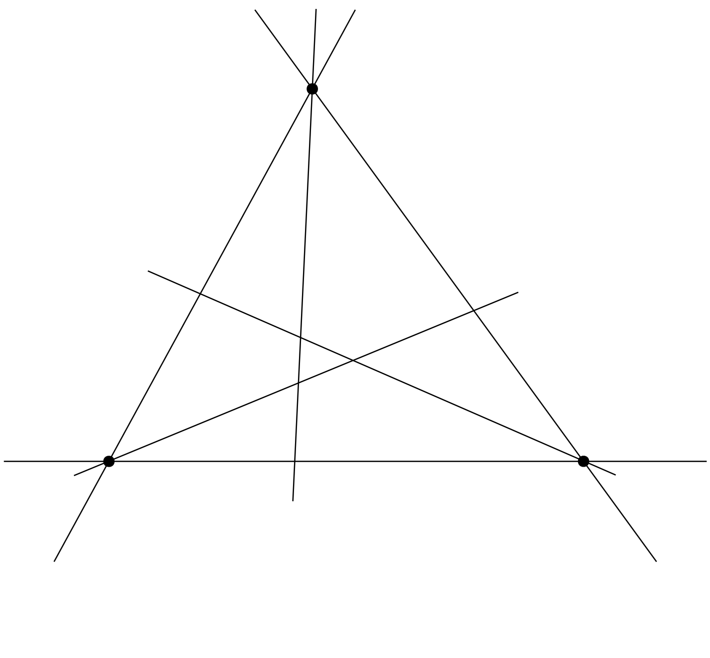
5.5.1 Example

We shall compute the Zeta function of ${S\times E^n}/{G_{d,n}}$ in the case when S has particularly nice arithemtic properties.

Let S be the $K3$ surface studied in [AOP02]. It can be defined as double cover of $\mathbb{P}^2$ branched along the union of six lines given by

$$XYZ(X+\lambda Y)(Y+Z)(Z+X)=0.$$

One can see that it is a $K3$ surface with an obvious non-symplectic involution.



The resolution of the singularities of that surface is obtained in the following way: firstly we blow up 3 triple points that defines 24 points on the double cover, then we blow up resulting variety at 15 double points.

The corresponding table 2.2 is equal to

$k \backslash j$	0	1
0	$\frac{1}{(1-T)(1-qT)^{19}\left(1-q^2T\right)}$	$\frac{1}{\left(1-\gamma_q qT\right)\left(1-\gamma_q \pi^2 T\right)\left(1-\gamma_q \bar{\pi}^2 T\right)}$
1	$\frac{1}{(1-T)^9(1-qT)^9}$	1

Table 5.15: $Z_{S,k,j}(T)$

From 2.7 we have the following formula for zeta function of the classical Borcea-Voisin threefold:

$$Z_q\left(\widetilde{S \times E / \mathbb{Z}_2}\right) = \left(\left(Z_{E,0,0}(T) \cdot Z_{E,1,0}\left(\sqrt{q} \cdot T\right)\right) \otimes \left(Z_{S,0,0}(T) \cdot Z_{S,1,0}\left(\sqrt{q} \cdot T\right)\right)\right) \times$$
$$\times \left(\left(Z_{E,0,1}(T) \cdot Z_{E,1,1}\left(\sqrt{q} \cdot T\right)\right) \otimes \left(Z_{S,0,1}(T) \cdot Z_{S,1,1}\left(\sqrt{q} \cdot T\right)\right)\right) =$$

Now according to table 5.13 we have three cases depending on number of fixed points of α_E defined over $\mathbb{F}_q$. Resulting zeta functions are summarized in the following table:

$Z_{E,1,0}$	$Z_q\left(\widetilde{S \times E / \mathbb{Z}_2}\right)$
$\frac{1}{(1-T)^4}$	$\frac{\left(1-a_q y_q qT + {y_q}^2 q^3 T^2\right)\left(1-a_q \pi^2 y_q T + \pi^4 {y_q}^2 q T^2\right)\left(1-a_q \bar{\pi}^2 y_q T + \bar{\pi}^4 {y_q}^2 q T^2\right)}{(1-T)(1-qT)^{56}\left(1-q^2T\right)^{56}\left(1-q^3T\right)}$
$\frac{1}{(1-T)^3(1+T)}$	$\frac{\left(1-a_q y_q qT + {y_q}^2 q^3 T^2\right)\left(1-a_q \pi^2 y_q T + \pi^4 {y_q}^2 q T^2\right)\left(1-a_q \bar{\pi}^2 y_q T + \bar{\pi}^4 {y_q}^2 q T^2\right)}{(1-T)(1-qT)^{47}(1+qT)^9\left(1+q^2T\right)^9\left(1-q^2T\right)^{47}\left(1-q^3T\right)}$
$\frac{1}{(1-T)^2(1+T+T^2)}$	$\frac{\left(1-a_q y_q qT + {y_q}^2 q^3 T^2\right)\left(1-a_q \pi^2 y_q T + \pi^4 {y_q}^2 q T^2\right)\left(1-a_q \bar{\pi}^2 y_q T + \bar{\pi}^4 {y_q}^2 q T^2\right)}{(1-T)(1-qT)^{38}\left(1-q^2T\right)^{38}\left(1+q^2T+q^4T^2\right)^9\left(1+qT+q^2T^2\right)^9\left(1-q^3T\right)}$

Table 5.16: Zeta function of $Y_{2,2}$

§ 5.6 Order 6

Consider $K3$ surface S_6 analysed in 5.4.1 and elliptic curve E_6 with the Weierstrass equation $y^2 = x^3 + 1$ together with a non-symplectic automorphism of order 6. Local zeta functions for both S_6 and E_6 are written in the following tables:

$k \backslash j$	0	1	2	3	4	5
0	$\frac{1}{(1-T)(1-qT)}$	$1-\alpha_q T$	1	1	1	$1-\overline{\alpha}_q T$
1	$\frac{1}{1-T}$	1	1	1	1	1
2	$\frac{1}{(1-T)^2}$	1	1	$\frac{1}{1-T}$	1	1
3	$\frac{1}{(1-T)^2}$	1	$\frac{1}{1-T}$	1	$\frac{1}{1-T}$	1
4	$\frac{1}{(1-T)^2}$	1	1	$\frac{1}{1-T}$	1	1
5	$\frac{1}{1-T}$	1	1	1	1	1

Table 5.17: $Z_{E_6,k,j}$

$k \backslash j$	0	1	2	3	4	5
0	$\frac{1}{(1-T)(1-qT)^{19}(1-q^2T)}$	$\frac{1}{1-\beta_q T}$	1	$\frac{1}{1-c_q qT}$	1	$\frac{1}{1-\overline{\beta}_q T}$
1	$\frac{1}{(1-T)^3(1-qT)^{18}}$	1	1	1	1	1
2	$\frac{1}{(1-T)^6(1-qT)^{15}}$	1	1	1	1	1
3	$\frac{1}{(1-T)^{10}(1-qT)^{10}}$	1	$1-\delta_q T$	1	$1-\overline{\delta}_q T$	1
4	$\frac{1}{(1-T)^{15}(1-qT)^6}$	1	1	1	1	1
5	$\frac{1}{(1-T)^{18}(1-qT)^3}$	1	1	1	1	1

Table 5.18: $Z_{S_6,k,j}$

Applying formula 2.7.1 we get

$$Z_q\left(\widetilde{S_6 \times E_6/\mathbb{Z}_6}\right) = \left[\left(\frac{1}{(1-T)(1-q\cdot T)} \cdot \frac{1}{(1-\sqrt[6]{q}\cdot T)} \cdot \frac{1}{(1-\sqrt[6]{q^2}\cdot T)^2} \cdot \frac{1}{(1-\sqrt[6]{q^3}\cdot T)^2} \cdot\right.\right.$$

$$
\cdot \frac{1}{(1-\sqrt[6]{q^4}\cdot T)^2}\cdot\frac{1}{(1-\sqrt[6]{q^5}\cdot T)}\Bigg)\otimes\Bigg(\frac{1}{(1-T)(1-q\cdot T)^{19}(1-q^2\cdot T)}\times
$$

$$
\times\frac{1}{(1-\sqrt[6]{q}\cdot T)^3(1-\sqrt[6]{q}\cdot q\cdot T)^{18}}\times\frac{1}{(1-\sqrt[6]{q^2}\cdot T)^6(1-\sqrt[6]{q^2}\cdot qT)^{15}}\times
$$

$$
\times\frac{1}{(1-\sqrt[6]{q^3}\cdot T)^{10}(1-\sqrt[6]{q^3}\cdot q\cdot T)^{10}}\times\frac{1}{(1-\sqrt[6]{q^4}\cdot T)^{15}(1-\sqrt[6]{q^4}\cdot q\cdot T)^6}\times
$$

$$
\times\frac{1}{(1-\sqrt[6]{q^5}\cdot T)^{18}(1-\sqrt[6]{q^5}\cdot q\cdot T)^3}\Bigg)\Bigg]\times\left[(1-\alpha_qT)\otimes\left(\frac{1}{1-\beta_qT}\right)\right]\times
$$

$$
\times\left[\left(\frac{1}{1-\sqrt[6]{q^3}\cdot T}\right)\otimes(1-\sqrt[6]{q^3}\cdot\delta_q\cdot T)\right]\times\left[\left(\frac{1}{1-\sqrt[6]{q^2}\cdot T}\cdot\frac{1}{1-\sqrt[6]{q^4}\cdot T}\right)\otimes\left(\frac{1}{1-c_q\cdot q\cdot T}\right)\right]\times
$$

$$
\times\left[\left(\frac{1}{1-\sqrt[6]{q^3}\cdot T}\right)\otimes(1-\sqrt[6]{q^3}\cdot\overline{\delta_q}\cdot T)\right]\times\left[(1-\overline{\alpha_q}\cdot T)\otimes\left(\frac{1}{1-\overline{\beta_q}\cdot T}\right)\right]=
$$

$$
=\frac{(1-\alpha_q\beta_qT)(1-\delta_qT)(1-\overline{\delta_q}T)(1-\overline{\alpha_q}\overline{\beta_q}T)}{(1-T)(1-qT)^{103}(1-q^2T)^{103}(1-q^3T)}.
$$

Using the same method we are able to compute zeta functions of higher dimensional quotients $\widetilde{S_6\times E_6^{n-1}/\mathbb{Z}_6}$. In the table below we collect results for $n=2,3,4$.

n	$Z_q\left(\widetilde{S_6\times E_6^{n-1}/\mathbb{Z}_6}\right)$
2	$\frac{(1-\alpha_q\beta_qT)(1-\delta_qT)(1-\overline{\delta_q}T)(1-\overline{\alpha_q}\overline{\beta_q}T)}{(1-T)(1-qT)^{103}(1-q^2T)^{103}(1-q^3T)}$
3	$\frac{1}{(1-T)(1-qT)^{340}\left(1-\bar{\alpha}_q{}^2\bar{\beta}_qT\right)\left(1-\alpha_q{}^2\beta_qT\right)\left(1-q^2T\right)^{1402}\left(1-q^3T\right)^{340}\left(1-q^2c_qT\right)^2\left(1-q^4T\right)}$
4	$\frac{\left(1-\alpha_q{}^3\beta_qT\right)\left(1-q^2\delta_qT\right)\left(1-q^2\overline{\delta}_qT\right)\left(1-\overline{\alpha}_q{}^3\overline{\beta}_qT\right)}{(1-T)(1-qT)^{868}\left(1-q^2T\right)^{9548}\left(1-q^2c_qT\right)\left(1-q^3c_qT\right)\left(1-q^3T\right)^{9548}\left(1-q^4T\right)^{868}\left(1-q^5T\right)}$

Table 5.19: Zeta function of $Y_{6,2}$, $Y_{6,3}$, $Y_{6,4}$

§ 5.7 More constructions

The results from previous chapters can be used to construct more interesting examples of higher dimensional Calabi-Yau manifolds. We shall present constructions involving Fermat quadric $K3$ surface and some $K3$ surfaces with purely inseparable automorphisms of order 6.

Let

$$F_4 := \left\{X_1^4 + X_2^4 + X_3^4 + X_4^4 = 0\right\} \subset \mathbb{P}^3$$

be the Fermat quartic defined over $\mathbb{Q}$ in $\mathbb{P}^3$. Consider a non-symplectic automorphism

$$\sigma_4 : \ F_4 \ni \left(X_1, X_2, X_3, X_4\right) \to \left(\zeta_4 X_1, X_2, X_3, X_4\right) \in F_4$$

of order 4. The fixed locus $\mathrm{Fix}(\sigma_4)$ consists of genus 3 curve $D := \left\{X_2^4 + X_3^4 + X_4^4 = 0\right\}$.

Since there are no isolated fixed points, we have the following theorem:

Theorem 5.7.1. *There exists a crepant resolution of*

$$\widetilde{F_4^n / \mathbb{Z}_4^{n-1}} \to F_4^n / \mathbb{Z}_4^{n-1}.$$

Consequently $F_{4,2n} := \widetilde{F_4^n / \mathbb{Z}_4^{n-1}}$ *is a Calabi-Yau* $2n$*-fold.*

The corresponding table 2.1 equals

$k \backslash j$	0	1	2	3
0	$(XY)^2 + 2 \cdot XY + 1$	$X^2 + 6 \cdot XY$	$7 \cdot XY$	$Y^2 + 6 \cdot XY$
1	$1 + 3(X+Y) + XY$	0	0	0
2	$1 + 3(X+Y) + XY$	0	0	0
3	$1 + 3(X+Y) + XY$	0	0	0

Table 5.20: $F_{F_4,k,j}$

Therefore by 2.3.3:

$$h^{p,q}\left(F_{4,2n}\right) = \Bigg(\Big((XY)^2 + XY + 1 + \left(\sqrt[4]{XY} + \sqrt[4]{(XY)^2} + \sqrt[4]{(XY)^3}\right) \cdot \left(1 + 3(X+Y) + XY\right)\Big)^n + $$
$$+ \left(X^2 + 6 \cdot XY\right)^n + \left(7 \cdot XY\right)^n + \left(Y^2 + 6 \cdot XY\right)^n\Bigg)[X^p Y^q].$$

Propositions 3.1, 4.1 of [CH07] and 3.2.1 may be used to construct more higher dimensional Calabi-Yau varieties.

Let S_6, S'_6 be $K3$ surfaces with non-symplectic automorphisms γ_6 and γ'_6 of order 6 such that only isolated points of $\mathrm{Fix}(\gamma_6)$ and $\mathrm{Fix}(\gamma'_6)$ are of type $p_{(3,4)}$. Example of such $K3$ surface was described in Table 1 of [Dil12c].

Theorem 5.7.2. *There exists a crepant resolution of*

$$\widetilde{S_6 \times S'_6/\mathbb{Z}_6} \to S_6 \times S'_6/\mathbb{Z}_6.$$

Consequently $\widetilde{S_6 \times S'_6/\mathbb{Z}_6}$ *is a Calabi-Yau* 4*-fold.*

Proof. The only singularity we need to check is the singularity coming from points of the type $p_{(3,4)}$ of surfaces S_6 and S'_6. These points produce singularity of type $\frac{1}{6}(3,4,3,2)$ which has a crepant resolution, see 5.3 and section 2.1.3:

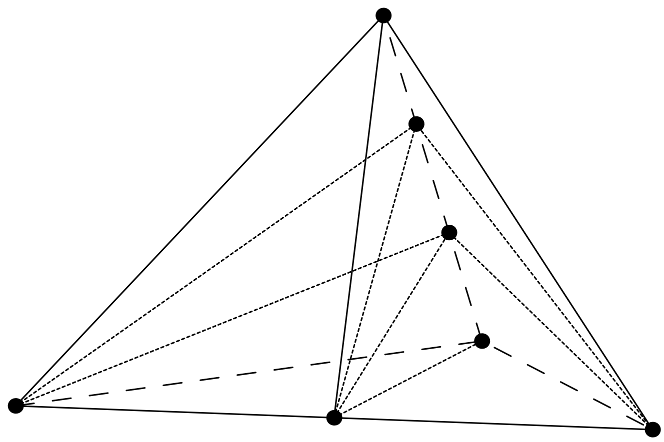


Figure 5.3: Crepant resolution of $\frac{1}{6}(3,4,3,2)$

The maps form $S_6 \times S'_6$ to the resolution are give in affine charts as

$$\left(\frac{x}{z}, \frac{t}{y^2}, y^3, z^2\right), \quad \left(\frac{x}{z}, \frac{t^2}{y}, \frac{y^2}{t}, z^2\right), \quad \left(\frac{x}{z}, t^3, \frac{y}{t^2}, z^2\right), \quad \left(\frac{z}{x}, \frac{t}{y^2}, y^3, x^2\right),$$
$$\left(\frac{z}{x}, \frac{t^2}{y}, \frac{y^2}{t}, x^2\right) \quad \text{or} \quad \left(\frac{z}{x}, t^3, \frac{y}{t^2}, x^2\right).$$

□

Remark 5.7.3. Similarly as before we can consider the following generalisations of the above constructions:

$$F_4 \times E_4^{n-1}/\mathbb{Z}_4^{n-1} \quad \text{and} \quad S_6 \times S'_6 \times E_6^{n-2}/\mathbb{Z}_6^{n-1},$$

where E_4 and E_6 denote elliptic curves with non-symplectic automorphisms of order 4 and 6, respectively. These quotients admits a crepant resolution of singularities by the same argument as in 5.3.1.

Chapter 6

Fibre products of elliptic surfaces

We discuss desingularized fiber product Calabi-Yau threefolds (introduced by Schoen) and give a method of computation of the Hodge numbers of fiberwise Kummer construction by using Chen-Ruan cohomology formula.

§ 6.1 Schoen's construction

Let (S_1, ϕ_1), (S_2, ϕ_2) be relatively minimal, rational elliptic surfaces with surjective maps $\phi_1 : S_1 \to \mathbb{P}^1$ and $\phi_2 : S_2 \to \mathbb{P}^1$. Consider $X := S_1 \times_{\mathbb{P}^1} S_2$ the fiber product of S_1 and S_2. In general X is not smooth. The singularities are the points (s_1, s_2) where s_1 and s_2 are singular points of the fibers of (S_1, ϕ_1), (S_2, ϕ_2) over a common point $u \in U := U_1 \cap U_2$, where U_1 and U_2 denote the images of the singular fibre of (S_1, ϕ_1), (S_1, ϕ_2) in $\mathbb{P}^1$.

Assume that all fibres over U are semi-stable. Since both S_1 and S_1 are rational with sections the fibre product X has trivial canonical bundle. The only singularities of the fiber product are ordinary double points (*nodes*), they correspond to pairs of singular points of fibers $(S_1)_u$ and $(S_2)_u$ for $u \in U$. In particular X is smooth iff $U = \emptyset$. A small resolution $\widetilde{X}$ of X is a Calabi-Yau threefold. This construction was introduced by C. Schoen ([Sch88]) who also gave a projectivity condition for $\widetilde{X}$.

M. Kapustka in [Kap09] gave extension of Schoen's contruction for reduced fibers i.e. **II**, **III**, **IV**.

Theorem 6.1.1 ([Kap09])**.** *Let S_1, S_2 be two rational elliptic surfaces with section. Then $X = S_1 \times_{\mathbb{P}^1} S_2$ admits small resolution iff all fiber of $S_1 \times_{\mathbb{P}^1} S_2$ are equal to $F \times \boldsymbol{I}_0$, $\boldsymbol{I}_n \times \boldsymbol{I}_m$, $\boldsymbol{III} \times \boldsymbol{I}_n$, $\boldsymbol{III} \times \boldsymbol{III}$, $\boldsymbol{IV} \times \boldsymbol{I}_n$, $\boldsymbol{II} \times \boldsymbol{II}$, where F denotes any fiber of an elliptic ruled surfaces.*

Following [CS12] assume that both fiber S_1, S_2 are semistable or they have the same type. Under this assumption X has a crepant resolution of singularities $\widetilde{X}$ which is a Calabi-Yau threefold.

6.1.1 Hodge numbers

In this section we discuss methods of computating of Hodge diamond of $\widetilde{X}$ (see [CS12]).

The Euler characteristic of $\widetilde{X}$ is the sum of Euler characteristics of singular fibers described in the following table

Fiber X_ξ	$\mathbf{I}_n \times \mathbf{I}_m$	$\mathbf{II} \times \mathbf{II}$	$\mathbf{III} \times \mathbf{III}$	$\mathbf{IV} \times \mathbf{IV}$
$e\left(X_\xi\right)$	$2mn$	6	12	24

Table 6.1: Euler characteristic of the singular fibers

Therefore

$$e\left(\widetilde{X}\right) = 2\left(\sum_{\xi \in U} t_1(\xi)t_2(\xi) + 2n_2 + 2n_3 + 3n_4\right), \tag{6.1.1}$$

where $t_1(\xi)$ (resp. $t_2(\xi)$) denote the number of components of the fiber $(S_1)_\xi$ (resp. $(S_2)_\xi$), n_2, n_3 and n_4 denote the number of singular fibers of type **II**, **III** and **IV**, respectively.

By the Shioda-Tate formula

(6.1.2)

$$h^{1,1}(\widetilde{X}) = d + 3 + \operatorname{rank}(\mathrm{MW}(S_1)) + \operatorname{rank}(\mathrm{MW}(S_2)) + \sum_{\xi \in U_1 \cup U_2} \left(\#\left(\text{components of } \widetilde{X}_\xi\right) - 1\right),$$

where $d = 1$ if S_1 and S_2 are isogeneous and $d = 0$ otherwise.

Using this formula Schoen gave examples of rigid Calabi-Yau threefolds as a self-fiber products of a rational elliptic surfaces with four semi-stable fibers (Beauville surface).

Merging formulas 6.1.1, 6.1.2 we can obtain the following

$$h^{1,2}(\widetilde{X}) = d + \#(U_1 \cup U_2) - 5 + \sum_{\xi \in U_1 \setminus U} (t_1(\xi) - 1) + \sum_{\xi \in U_2 \setminus U} (t_2(\xi) - 1).$$

§ 6.2 Kummer fibrations

Every birational map $\alpha_i : S_i \to S_i$ preserving fibration $\pi_i : S_i \to \mathbb{P}_i$ is an automorphism (see [IS96]). For a pair of $\alpha_i : S_i \to \mathbb{P}^1$ $(i = 1, 2)$ of automorphisms acting fiberwise we can consider also

$$\alpha := \alpha_1 \times_{\mathbb{P}^1} \alpha_2 : X_1 \times_{\mathbb{P}^1} X_2 \to X_1 \times_{\mathbb{P}^1} X_2$$

which lifts to an automorphisms $\alpha : X \to X$. If α preserves the canonical form on X and the quotient ${}^{X}\!/_{\alpha}$ admits an a crepant resolution of singularities then we again get Calabi-Yau threefold, we can use orbifold formula to compute its Hodge numbers.

Schoen studied the most natural case of construction, with quotient of X by the natural involution (fiberwise Kummer construction).

Let S_i be a rational elliptic surface with section $b_i : \mathbb{P}^1 \mapsto S_i$ for $i = 1, 2$. Suppose S_1, S_2 admit only singular fibers of type $\mathbf{I}_n$, $\mathbf{II}$, $\mathbf{III}$ or $\mathbf{IV}$. On the fiber product $X := S_1 \times_{\mathbb{P}^1} S_2$ we get an induced section $b = (b_1, b_2) : \mathbb{P}^1 \mapsto X$. Now consider involutions

$$i_k : S_k \ni x \mapsto b_k(\pi_k(x)) - x \in S_k,$$

where π_k denotes projection onto k-th coordinate, for $k = 1, 2$. Let us define involution

$$i : X \ni (x_1, x_2) \mapsto \big(i_1(x_1), i_2(x_2)\big) \in X.$$

Observe that i is well defined because on each fiber we have a group structure.

Theorem 6.2.1 ([Kap09]). *There exists a crepant resolution of singularities*

$$Y \mapsto {}^{X}\!/_{\langle i \rangle}.$$

Consequently Y is a Calabi-Yau threefold (not necessarily projective).

Moreover if X does not have singular fiber of type $\boldsymbol{I}_1 \times \boldsymbol{I}_n$, then Y admits projective crepant resolution of singularities.

Kapustka gave a method of computing Hodge numebrs using presentation of S_1, S_2 as double quartics (see [Kap09]). We shall describe more direct approach based on orbifold's formula (see section 2.3). Let $\widetilde{X}$ be a Calabi-Yau resolution of X, the involution i lifts to an involution on $\widetilde{X}$.

Since we consider only small resolutions all components of $\mathrm{Fix}(i)$ contained in fibers are rational hence we do not affect $H^{1,2}_{\mathrm{orb}}(Y)$. From 2.3 it follows that

$$H^{1,2}(Y) = H^{1,2}\left(\widetilde{X}\right)^{\langle i \rangle} + \bigoplus_{C \in \Lambda(\mathrm{Fix}(i))} H^{0,1}(C) = H^{1,2}\left(X\right)^{\langle i \rangle} + \bigoplus_{C \in \Lambda(\widetilde{C_1 \times_{\mathbb{P}1} C_2})} g(C),$$

where $C_1 := \mathrm{Fix}(i_1), C_2 = \mathrm{Fix}(i_2)$ and $\Lambda(\widetilde{C_1 \times_{\mathbb{P}^1} C_2})$ denotes the set of connected components of the normalization $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$ of the fiber product $C_1 \times_{\mathbb{P}^1} C_2$.

From Hurwitz formula

$$\bigoplus_{C\in\Lambda(\widetilde{C_1\times_{\mathbb{P}^1}C_2})}(2-2g(C)) = 2\cdot 16 - \sum_{P\in\widetilde{C_1\times_{\mathbb{P}^1}C_2}}(e_P-1)$$

where e_P denotes multiplicity of P. Therefore in order to find $\bigoplus_{C\in\Lambda(\widetilde{C_1\times_{\mathbb{P}^1}C_2})} g(C)$ it suffices to compute

- number of components of $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$,
- ramification index of any point $P \in \widetilde{C_1 \times_{\mathbb{P}^1} C_2}$.

Sections C_1 i C_2 can be presented by using monodromy representation:

Theorem 6.2.2 ([Mir95]). *Let $B = \{b_1, b_2, \dots, b_n\} \subset \mathbb{P}^1$ be a finite set of points. Then there is a $1-1$ correspondence*

$$\left\{\begin{array}{l}\text{Isomorphism classes of holomorphic}\\ \text{maps } F:\ X\to\mathbb{P}^1 \text{ of degree } d\\ \text{whose branch points lie in } B\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{Conjugacy classes of } n\text{-tuples } (\sigma_1,\dots,\sigma_n)\\ \text{of permutations in } \Sigma_d \text{ such that } \sigma_1\dots\sigma_n = 1\\ \text{and the subgroup generated by the } \sigma_i\text{'s}\\ \text{is transitive}\end{array}\right\}$$

Adding some unramified points we can assume that they are given as sequences of n points $b_1, \dots, b_n \in \mathbb{P}^1$ and permutations $\sigma_1^{(i)}, \sigma_2^{(i)}, \dots, \sigma_n^{(i)} \in \Sigma_{d_i}$ for $i = 1, 2$, where d_i is the degree of map $C_i \to \mathbb{P}^1$ and

$$\sigma_1^{(i)} \dots \sigma_n^{(i)} = 1.$$

We do not assume that the group $\langle\sigma_1^{(i)} \dots \sigma_n^{(i)}\rangle$ is transitive as the curve C_i can be reducible.

Now forming the product representation $\sigma_i = \sigma_1^{(i)} \times \sigma_1^{(i)}, \dots, \sigma_n = \sigma_n^{(i)} \times \sigma_n^{(i)}$ we get monodromy representations of $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$ which allow us to compute components of $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$ their genera and consequently the sum of genera.

6.2.1 Ramification index of point $P \in \widetilde{C_1 \times_{\mathbb{P}^1} C_2}$

If i_1, i_2 are involutions then curves C_1 and C_2 are fourfold covers of $\mathbb{P}^1$, if moreover they are reducible (which happens in particular for standard Kummer fibration) then the sum of genera of components of $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$ can be computed form the analysis of behaviour of involution on singular fiber.

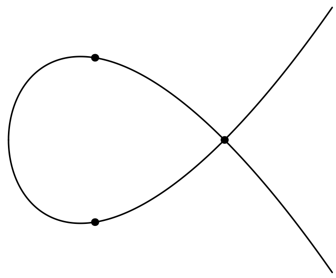
We start with a careful study of the action of the involution i on singular fibers of S_1, S_2.

Lemma 6.2.3 ([Kap09])**.** *Assume that S is a rational elliptic surface with chosen 0 section with reduced fibers. Let i be the involution of the form*

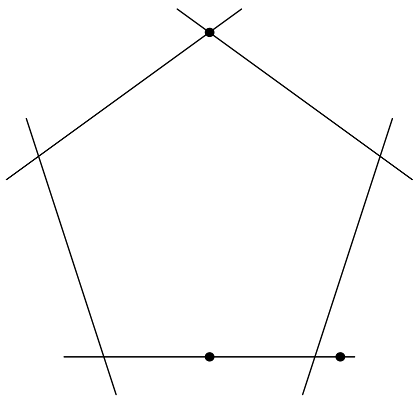
$$i\colon\ x \mapsto b - x,$$

where b is a section of S. Let F be a singular fiber of S. Then we have the following possibilities:

- *The fiber F is of type $\boldsymbol{I}_1$. Then i acts on F by symmetry with three fixed points*

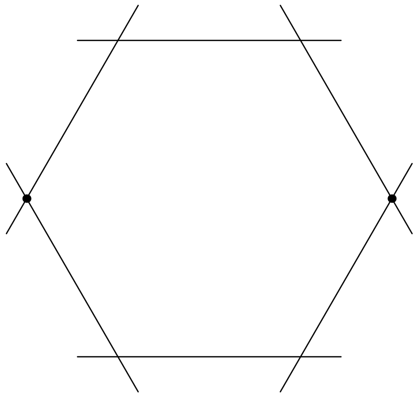


- *The fiber F is of type $\boldsymbol{I}_{2k+1}$, where $k \geq 1$. Then i acts on one of the components of F by symmetry with two fixed points outside singularities of F and interchanges the respective pairs of the remaining components. The singularity opposite to the component on which i acts is fixed*

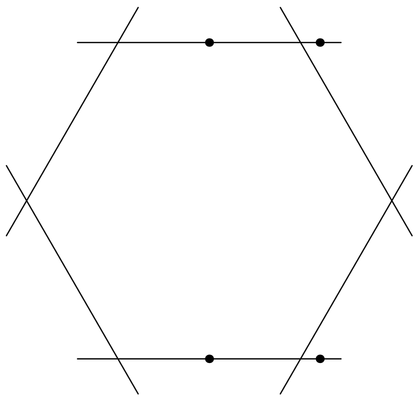


- *The fiber F is of type $\boldsymbol{I}_{2k}$, where $k \geq 1$. Then we have one of the following cases:*

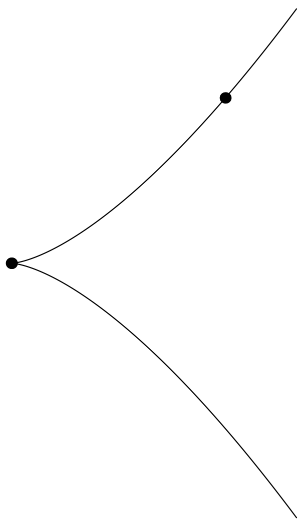
  1. *The involution i has two fixed points – two opposite singularities of F and interchanges pairs of components of F.*



2. *The involution i acts on two opposite components of F by symmetry, interchanges respective remaining pairs of components – i has 4 fixed points outside singularities of F.*

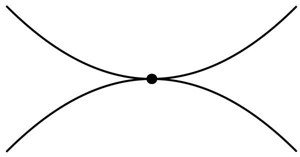


- *The fiber F is of type $\boldsymbol{II}$. Then i has two fixed point together with singular point of F.*

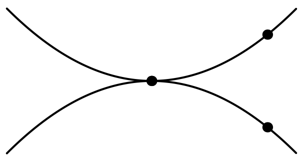


- *The fiber F is of type **III**. Then we have one of the following cases:*

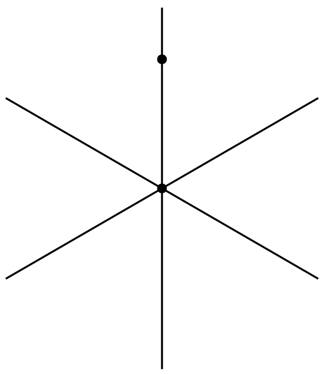
  1. *The involution i fixes only singular point and interchanges two components of F.*



  2. *The involution i acts on both components fixing singular point and one more point on each component – in total three fixed points.*



- *The fiber F is of type **IV**. Then i fixes the triple singular point, interchanges two of the components of the fiber and acts on third component fixing one more point – in total two fixed points.*



6.2.2 Components of the fixed locus of Fix(i)

We shall give a more direct description of components of Fix(i) in the case when i_1, i_2 are non-symplectic involutions. In this case both curve (possible reducible) C_1, C_2 give $4 : 1$ coverings of $\mathbb{P}^1$.

$$\begin{tikzcd} C_1 \arrow[dr, "4:1"'] & & C_2 \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Let us compute the number of irreducible components of $C_1 \times_{\mathbb{P}^1} C_2$. Assume that D_1 and D_2 are irreducible components of C_1 and C_2, respectively. Consequently, for each irreducible connected component $C \subseteq D_1 \times_{\mathbb{P}^1} D_2$ we have the following diagram

$$\begin{tikzcd} & C \arrow[dl] \arrow[dr] & \\ D_1 \arrow[dr] & & D_2 \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd}$$

where $D_k \to \mathbb{P}^1$ is a covering of degree at most 4, for $k = 1, 2$.

Observation 6.2.4. *We have the following cases*

1. *If $D_1 \to \mathbb{P}^1$, $D_2 \to \mathbb{P}^1$ are $a : 1$ and $b : 1$ coverings, where $\gcd(a, b) = 1$*

$$\begin{tikzcd} & C \arrow[dl, "b:1"'] \arrow[dr, "a:1"] & \\ D_1 \arrow[dr, "a:1"'] & & D_2 \arrow[dl, "b:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "a:1"'] \arrow[dr, "b:1"] & \\ D_1 \arrow[dr, "b:1"'] & & D_2 \arrow[dl, "a:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

then $C \simeq D_1 \times_{\mathbb{P}^1} D_2$ and hence we get one component.

2. *If $D_1 \to \mathbb{P}^1$, $D_2 \to \mathbb{P}^1$ are $2 : 1$ coverings, then we have two possible diagrams*

$$\begin{tikzcd} & C \arrow[dl, "1:1"'] \arrow[dr, "1:1"] \arrow[dd, dashed, "2:1"] & \\ D_1 \arrow[dr, "2:1"'] & & D_2 \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "2:1"'] \arrow[dr, "2:1"] \arrow[dd, dashed, "4:1"] & \\ D_1 \arrow[dr, "2:1"'] & & D_2 \arrow[dl, "2:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

then

- *$D_1 \times_{\mathbb{P}^1} D_2$ has two components, each of them corresponds to the first diagram; in this case $D_1 \simeq D_2$ (as $\mathbb{P}^1$-curves) or*
- *$D_1 \times_{\mathbb{P}^1} D_2$ is irreducible.*

3. *If $D_1 \to \mathbb{P}^1$, $D_2 \to \mathbb{P}^1$ are $3:1$ coverings, then we have three possible diagrams*

$$\begin{tikzcd} & C \arrow[dl, "1:1"'] \arrow[dr, "1:1"] \arrow[dd, dashed, "3:1"] & \\ D_1 \arrow[dr, "3:1"'] & & D_2 \arrow[dl, "3:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "2:1"'] \arrow[dr, "2:1"] \arrow[dd, dashed, "6:1"] & \\ D_1 \arrow[dr, "3:1"'] & & D_2 \arrow[dl, "3:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "3:1"'] \arrow[dr, "3:1"] \arrow[dd, dashed, "9:1"] & \\ D_1 \arrow[dr, "3:1"'] & & D_2 \arrow[dl, "3:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

then

- *$D_1 \times_{\mathbb{P}^1} D_2$ has three components, each of them corresponds to the first diagram; in this case $D_1 \simeq D_2$ and the covering $D_1 \to \mathbb{P}^1$ is cyclic or*
- *$D_1 \times_{\mathbb{P}^1} D_2$ has two components, one of them corresponds to the first diagram and the later to the second one; in this case $D_1 \simeq D_2$ but the covering $D_1 \to \mathbb{P}^1$ is not cyclic, or*
- *$D_1 \times_{\mathbb{P}^1} D_2$ is irreducible.*

4. *If $D_1 \to \mathbb{P}^1$, $D_2 \to \mathbb{P}^1$ are $2:1$ and $4:1$ coverings, then we have two possible diagrams (and two symmetric one)*

$$\begin{tikzcd} & C \arrow[dl, "2:1"'] \arrow[dr, "1:1"] \arrow[dd, dashed, "4:1"] & \\ D_1 \arrow[dr, "2:1"'] & & D_2 \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "4:1"'] \arrow[dr, "2:1"] \arrow[dd, dashed, "8:1"] & \\ D_1 \arrow[dr, "2:1"'] & & D_2 \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

then

- *$D_1 \times_{\mathbb{P}^1} D_2$ has two components, each of them corresponds to the first diagram and $\pi_2 : D_2 \to \mathbb{P}^1$ factors through $\pi_1 : D_1 \to \mathbb{P}^1$ or*
- *$D_1 \times_{\mathbb{P}^1} D_2$ is irreducible.*

5. *If $D_1 \to \mathbb{P}^1$, $D_2 \to \mathbb{P}^1$ are $4:1$ coverings, then we have three possible diagrams*

$$\begin{tikzcd} & C \arrow[dl, "1:1"'] \arrow[dr, "1:1"] \arrow[dd, dashed, "4:1"] & \\ D_1 \arrow[dr, "4:1"'] & & D_2 \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "2:1"'] \arrow[dr, "2:1"] \arrow[dd, dashed, "8:1"] & \\ D_1 \arrow[dr, "4:1"'] & & D_2 \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, "4:1"'] \arrow[dr, "4:1"] \arrow[dd, dashed, "16:1"] & \\ D_1 \arrow[dr, "4:1"'] & & D_2 \arrow[dl, "4:1"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

then

- $D_1 \times_{\mathbb{P}^1} D_2$ *has four components, each of them corresponds to the first diagram or*
- $D_1 \times_{\mathbb{P}^1} D_2$ *has three components, one of them corresponds to the second diagram and the later two to the first diagram or*
- $D_1 \times_{\mathbb{P}^1} D_2$ *has two irreducible components, each of them corresponds to the second diagram or*
- $D_1 \times_{\mathbb{P}^1} D_2$ *is irreducible.*

In cases 4 *and* 5 *we can give only necessary conditions in terms of ramification indices. Finally note that a pair of points in fibers over the same* $b \in \mathbb{P}^1$ *with ramification indices* e_1 *and* e_2*, respectively give rise to* d *points in* $\widetilde{D_1 \times_{\mathbb{P}^1} D_2}$ *with ramification indices* $\frac{e_1 e_2}{d}$*, where* $d = \gcd(e_1, e_2)$.

6.2.3 Examples

Beauville surface $\mathbf{I}_3\mathbf{I}_3\mathbf{I}_3\mathbf{I}_3$

Consider rational elliptic surface $S \to \mathbb{P}^1(t)$ given in Weierstrass form

$$y^2 = 4x^3 - 3(8t^3 + 1)x - 8t^6 - 20t^3 + 1.$$

The discriminant of the right hand side is equal to

$$432t^3 \left(2t + 1 + \sqrt{-3}\right)^3 (t-1)^3 \left(-2t - 1 + \sqrt{-3}\right)^3.$$

Since $4x^3 - 3(8t^3 + 1)x - 8t^6 - 20t^3 + 1$ is irreducible, we get a 3-section, which is not a cyclic covering. Therefore the self fiber product has 5 irreducible components; three of them came from case 1 another two from case 3.

Since the ramification index of fixed points in $\mathbf{I}_3 \times \mathbf{I}_3$ are equal to $1, 1, 1, 1, 2, 2, 2, 2, 2, 2$. From 6.2 it follows that

$$10 - 2 \sum_{C \in \Lambda(C_1 \times_{\mathbb{P}^1} C_2)} g(C) = 2 \cdot 16 - 4 \cdot (6(2-1) + 4(1-1)) = 8,$$

so $\displaystyle\sum_{C \in \Lambda(C_1 \times_{\mathbb{P}^1} C_2)} g(C) = 1.$

We can get more precise description of the geometry of the fixed locus of involution using a monodromy argument. The monodromy representation (computed with function `monodromy` from MAPLE package `algcurves`) is presented in the following table

0	1	ζ_3	ζ_3^2
$(2,3)$	$(1,3)$	$(1,2)$	$(1,3)$

Table 6.2: Monodromy of Beauville surface $\mathbf{I}_3\mathbf{I}_3\mathbf{I}_3\mathbf{I}_3$

Therefore the monodromy representation in $\Sigma_4 \times \Sigma_4 \subset \Sigma_{16}$ is given by

0	$(2,3)(5,9)(6,11)(7,10)(12,8)(14,15)$
1	$(1,11)(2,10)(3,9)(4,12)(5,7)(13,15)$
ζ_3	$(1,6)(2,5)(3,7)(4,8)(9,10)(13,14)$
ζ_3^2	$(1,11)(2,10)(3,9)(4,12)(5,7)(13,15)$

The group G generated by the above four permutations in Σ_{16} acts on $\{1, 2, \dots, 16\}$ in a non-transitive way. In fact the set $\{1, 2, \dots, 16\}$ decomposes into a union of five sets

$$\{1, 2, \dots, 16\} = \{1, 6, 11\} \cup \{2, 3, 5, 7, 9, 10\} \cup \{4, 8, 12\} \cup \{13, 14, 15\} \cup \{16\}.$$

The group G preserves each of theses five sets and induces transitive action. Consequently $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$ decomposes into 5 connected components with degrees of the map to $\mathbb{P}^1$ equals 3, 6, 3, 3, 1. Over the four branch point $\widetilde{C_1 \times_{\mathbb{P}^1} C_2}$ we have six ramification points with indices 2. The number of ramification points on the five components equals 4, 12, 4, 4, 0. Hence the genera equals 0, 1, 0, 0, 0.

Remark 6.2.5. The advantage of this monodromy based approach is that it requires only knowledge of a Weierstrass equation.

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